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Comment on Measuring the Size of the Shadow Economy Using a Dynamic General Equilibrium Model with Trends

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LINK TO ABSTRACT

Various approaches have been proposed in the literature to estimate the size of the informal or shadow economy. Following Ceyhun Elgin (2020), the approaches can be classified as direct, indirect, and model approaches. Direct approaches provide estimates using direct methods as surveys, interviews, and questionnaires on households or firms. Indirect approaches infer the size of the shadow economy using data on economic indicators, such as currency demand or electricity consumption, or on the discrepancy between actual and registered labor force or between national income and expenditure statistics. Model approaches generally rely on the use of a theoretical model, such as the dynamic general equilibrium (DGE) model and the multiple indicators multiple causes (MIMIC) model, which is based on the use of a specific structural equation model.

As Elgin et al. (2021) argue, the DGE method has several limitations, including the need of a base-year estimate of the size of shadow production from an external source and its reliance on strong and somewhat arbitrary assumptions on the relationship between the productivities in the shadow and formal sectors. Mario Solis-Garcia and Yingtong Xie (2018) propose a DGE method for measur-

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ing the size and dynamics of the shadow economy. They apply the method to a set of Latin American and Asian countries. They set up a two-sector dynamic deterministic general equilibrium model with four different exogenous trends, and use the restrictions imposed by the model's balanced growth to endogenize the shadow productivity trend. This method would avoid imposing an arbitrary assumption about productivities in the shadow and formal sectors.

Unfortunately, the method proposed by Solis-Garcia and Xie (2018) does not serve its purpose of computing the ratio of shadow to formal production. The reason is that the balanced-growth assumption entails that formal and shadow output grow at the same rate, and hence their ratio must be constant at all times and equal to its base-year value, which is taken as given (from Schneider et al. 2010). The problem should have become apparent in the numerical simulations made by the authors. They did not notice the problem because of an error in the formula for computing the growth rate of the productivity shock in the shadow economy. This error propagated to other variables of the model and, eventually, led to a non-constant ratio of shadow to formal output, which gave a sense of plausibility to the simulation results.

The model

The method proposed by Solis-Garcia and Xie (2018) relies on a two-sector (formal and shadow) deterministic DGE model.

Setup

The economy is inhabited by a representative infinitely lived household who solves the utility maximization problem:

$$\max_{\left\{C_t, N_t, N_{F,t}, N_{S,t}, K_{t+1}, X_t\right\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\phi \Gamma_{H,t} N_t^{1+\chi}}{1+\chi} \right)$$

$$\text{subject to:} \quad C_t + \Gamma_{A,t} X_t = \left(1-\tau_t\right) Y_{F,t} + \left(1-\rho \hat{s} \tau_t\right) Y_{S,t} \ ,$$

$$K_{t+1} = (1-\delta) K_t + X_t \ ,$$

$$N_t = N_{F,t} + N_{S,t} \ ,$$

where formal output, $Y_{F,t}$, and shadow output, $Y_{S,t}$, are given by

$$Y_{F,t} = K_t^{\alpha}\left(\Gamma_{F,t}N_{F,t}\right)^{1-\alpha}, \tag{1}$$

$$Y_{S,t} = \left(\Gamma_{S,t}N_{S,t}\right)^{\eta}. \tag{2}$$

In the maximization problem, C_t is consumption, N_t is total hours worked, X_t is investment, $\tau_t \in [0,1)$ is the income tax rate, K_t is the stock of capital, $N_{F,t}$ and $N_{S,t}$ are hours worked in formal and shadow production, and total output is $Y_t = Y_{F,t} + Y_{S,t}$. The model includes four exogenous permanent productivity shocks: in the household's choice of hours worked, $\Gamma_{H,t}$; the production of investment goods, $\Gamma_{A,t}$; the hours worked in formal production, $\Gamma_{F,t}$; and the hours worked in shadow production, $\Gamma_{S,t}$. The parameter $\beta \in (0,1)$ is the discount factor, $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution, $\phi > 0$ is the disutility of labor, $\chi \geq 0$ is the inverse of the Frisch elasticity of labor supply, $\delta \in (0,1)$ is the depreciation rate of capital, $\rho \in [0,1)$ is the probability of a tax audit, $\hat{s} > 1$ is a tax surcharge, $\alpha \in (0,1)$ is the capital share in formal production, and $\eta > 0$ is the elasticity of work time in shadow production.

The government taxes income at a rate τ_t and uses tax revenues to fund a stream of non-productive expenditure G_t. The informal sector avoids taxation unless caught by a tax audit, which occurs with probability ρ, in which case the government imposes a tax surcharge $\hat{s}$. Thus the government's budget constraint is

$$G_t = \tau_t Y_{F,t} + \rho\hat{s}\tau_t Y_{S,t}.$$

Balanced growth path

Let $g_{Z,t} = Z_t / Z_{t-1}$ denote the (gross) growth rate of a variable Z, so that $Z_t = Z_0\prod_{j=1}^{t} g_{Z,j}$ $(t > 0)$, and let $g_{i,t} = \Gamma_{i,t} / \Gamma_{i,t-1}$ denote the growth rate of the exogenous productivity shock $i \in \{H, A, F, S\}$, so that $\Gamma_{i,t} = \Gamma_{i,0}\prod_{j=1}^{t} g_{i,j}$ $(t > 0)$.

Solis-Garcia and Xie (2018) use their equilibrium conditions (expressions 3.5 through 3.15 in their article) to calculate the balanced-growth rates, which are derived from their conditions (expressions 4.1–4.10 in their article), which we reproduce for reference:

$$g_C = g_A g_X = g_G = g_Y \tag{3}$$

$$g_K = g_X \tag{4}$$

$$g_N = g_{N_F} = g_{N_S} \tag{5}$$

$$g_A = g_{Y_F} g_K^{-1} \tag{6}$$

$$g_H g_N^{\chi} = g_{Y_F}^{1-\sigma} g_{N_F}^{-1} \tag{7}$$

$$g_{Y_S} g_{N_S}^{-1} = g_{Y_F} g_{N_F}^{-1} \tag{8}$$

$$g_{Y_F} = g_K^{\alpha} \left(g_F g_{N_F} \right)^{1-\alpha} \tag{9}$$

$$g_{Y_S} = \left(g_S g_{N_S} \right)^{\eta} \tag{10}$$

$$g_Y = g_{Y_F} = g_{Y_S} \tag{11}$$

$$g_A = g_{P_X} \tag{12}$$

Here, P_X is the decentralized price of investment goods, and income tax rates are assumed to be stationary, $g_\tau = 1$.

There are 14 equalities in the system (3)–(12).[3] As Eqs. (6) and (8) can be derived from (3), (4), (5) and (11), we can delete them. The resulting system has 12 equalities, so 12 of the 15 growth rates can be solved for as functions of the three remaining ones. This means that of the growth rates of the exogenous productivity shocks, one can be expressed as a function of the remaining three, namely, g_S is chosen to be a function of g_H , g_F and g_A .

Computation of the ratio of shadow to formal output

Using their balanced-growth conditions, Solis-Garcia and Xie (2018) compute the ratio of shadow to formal production, $Y_{[S/F],t} = Y_{S,t} \big/ Y_{F,t}$ by following these steps:

Step 1. Obtain data for the stock of capital, K_t , formal work time, $N_{F,t}$, and formal output, $Y_{F,t}$. Use them to calculate the time series of the observed growth rates

3. Taking logarithms in (3)–(12), the resulting system is linear in the logarithms of the growth rates so it can be readily discussed and solved.

$\hat{g}_{K,t}$, $\hat{g}_{N_F,t}$, and $\hat{g}_{Y_F,t}$.
Step 2. Select a base year t_0 and the values of the parameters α, σ, χ, ρ, and $\hat{s}$.
Step 3. Calibrate the value of the elasticity of shadow work, η.
Step 4. Calculate the base-year value of shadow work time, N_{S,t_0}, taking as given the base-year formal production, Y_{F,t_0}, from actual data, and the base-year ratio of shadow to formal production, $Y_{[S/F],t_0}$, from Schneider et al. (2010). Compute the growth rates of the productivity shock in shadow production, $g_{S,t}$, from the observed growth rates and then the time series of the productivity shock, $\Gamma_{S,t}$.
Step 5. Compute the time series of work time in shadow production $N_{S,t}$ using that $g_{N_S,t} = g_{N_F,t} = \hat{g}_{N_F,t}$. This equation follows from the balanced-growth condition (5) and the assumption that the growth rate of formal work time, $g_{N_F,t}$, is equal to its observed value, $\hat{g}_{N_F,t}$.
Step 6. Compute the time series of shadow output $Y_{S,t} = \left(\Gamma_{S,t} N_{S,t}\right)^{\eta}$ and then the time series of the ratio of shadow to formal output $Y_{[S/F],t} = Y_{S,t} \big/ Y_{F,t}$.

Why does this method fail?

Solis-Garcia and Xie (2018) take as given the base-year value of formal production, Y_{F,t_0}, from actual data, and the base-year value of the ratio of shadow to formal production, $Y_{[S/F],t_0}$, from Friedrich Schneider, Andreas Buehn, and Claudio Montenegro (2010). Hence, the initial value of shadow output is determined by $Y_{S,t_0} = Y_{[S/F],t_0} Y_{F,t_0}$. The balanced-growth condition (11)—Eq. (4.9) in Solis-Garcia and Xie (2018)—says the growth rate of shadow output is equal to the growth rate of formal (and of total) output, which is computed from actual data, $g_{Y_S,t} = g_{Y_F,t} = \hat{g}_{Y_F,t}$. The immediate consequence is that the ratio of shadow to formal output must be constant and equal to its base-year value at every time,

$$Y_{[S/F],t} = \frac{Y_{S,t}}{Y_{F,t}} = \frac{Y_{S,t_0} \ \Pi_{j=t_0+1}^{t} g_{Y_S,j}}{Y_{F,t_0} \ \Pi_{j=t_0+1}^{t} g_{Y_F,j}} = \frac{Y_{S,t_0}}{Y_{F,t_0}} = Y_{[S/F],t_0}, \text{if } t > t_0,$$

or

$$Y_{[S/F],t} = \frac{Y_{S,t}}{Y_{F,t}} = \frac{Y_{S,t_0} \Big/ \prod_{j=t+1}^{t_0} g_{Y_{S,j}}}{Y_{F,t_0} \Big/ \prod_{j=t+1}^{t_0} g_{Y_{F,j}}} = \frac{Y_{S,\,t_0}}{Y_{F,t_0}} = Y_{[S/F],\,t_0}, \text{if } t_0 > t.$$

The method, therefore, generates a constant ratio of shadow to formal production and, therefore, is useless to compute the time series of this ratio. Despite its obviousness, the problem with this method has so far gone unnoticed, probably because of the relative complexity of its implementation.

The question that arises now is why the simulation results in Solis-Garcia and Xie (2018) do not display a constant ratio of shadow to formal production as they should. The reason is that there is an error in their expression (4.14) for g_S which causes the simulated ratio to be variable. Appendix A shows that the true expression for g_S is

$$g_S = g_H^{-\frac{1-\eta}{(\sigma+\chi)\eta}} \, g_A^{-\frac{\alpha[(1+\chi)-(1-\sigma)\eta]}{(1-\alpha)(\sigma+\chi)\eta}} \, g_F^{\frac{(1+\chi)-(1-\sigma)\eta}{(\sigma+\chi)\eta}}. \tag{13}$$

In contrast, the expression (4.14) in Solis-Garcia and Xie (2018) is

$$g_S = g_H^{-\frac{1+\eta}{(\sigma+\chi)\eta}} \, g_A^{-\frac{\alpha[(1+\chi)+(1-\sigma)\eta]}{(1-\alpha)(\sigma+\chi)\eta}} \, g_F^{\frac{1+\chi+(1-\sigma)\eta}{(\sigma+\chi)\eta}},$$

where the differences with (13) are marked in red.

The error in the computation of the growth rate $g_{S,t}$ translates to the productivity shock in shadow production $\Gamma_{S,t} = \Gamma_{S,t_0} \prod_{j=t_0+1}^{t} g_{S,j}$ if $t > t_0$ (or $\Gamma_{S,t} = \Gamma_{S,t_0} \Big/ \prod_{j=t+1}^{t_0} g_{S,j}$ if $t_0 > t$), which in turn translates to shadow production $Y_{S,t} = \left(\Gamma_{S,t} N_{S,t}\right)^{\eta}$ and, finally, to the ratio of shadow to formal output $Y_{[S/F],t} = Y_{S,t} \Big/ Y_{F,t}$. As a result, the ratio of shadow to formal output is variable in their simulation results, which gives the proposed method an appearance of validity. Using the correct formula for g_S in the Matlab scripts made available by the authors, the simulated ratio $Y_{[S/F],t}$ is constant, as expected.[4]

4. The original MATLAB files are available at Solis-Garcia's website (link). The files, with the error corrected, that reproduce the simulated shadow and formal output and its ratio are available from the journal website (link).

Conclusions

The DGE method proposed in Solis-Garcia and Xie (2018) was intended to overcome some of the limitations of previous approaches. Unfortunately, it does not serve its purpose of estimating the ratio of shadow to formal production. The goal is valuable, but more research will be needed to achieve it.

Appendix A

Derivation of g_S

In this Appendix we show that the expression (4.14) for the growth rate of the productivity shock in shadow production, g_S, in Solis-Garcia and Xie (2018) is wrong, and we compute its true value. To show the scope and the source of the error, we provide a detailed derivation starting from the restrictions imposed by the model's balanced-growth conditions (3)–(12), which are expressions 4.1–4.10 in Solis-Garcia and Xie (2018).

From (3) and (4) we get

$$g_Y = g_A g_K. \tag{14}$$

Using (11) and (14), we have

$$g_Y = g_{Y_F} = g_{Y_S} = g_A g_K, \tag{15}$$

whereas (5) states that $g_{N_F} = g_{N_S} = g_N$. Using (15) and (5), we substitute g_{Y_F} and g_{Y_S} with $g_A g_K$, and substitute g_{N_F} and g_{N_S} with g_N, into Eqs. (7), (9) and (10), and get

$$g_H g_N^{1+\chi} = g_A^{1-\sigma} g_K^{1-\sigma}, \tag{16}$$

$$g_A g_K^{1-\alpha} = g_N^{1-\alpha} g_F^{1-\alpha}, \tag{17}$$

$$g_A g_K = g_S^{\eta} g_N^{\eta}. \tag{18}$$

Solving for g_N in (17), we get that

$$g_N = g_A^{\frac{1}{1-\alpha}} \; g_K \; g_F^{-1}. \tag{19}$$

Plugging this expression into (16), we have

$$g_A^{1-\sigma} g_K^{1-\sigma} = g_H \left(g_A^{\frac{1}{1-\alpha}} \; g_K \; g_F^{-1} \right)^{1+\chi} = g_H \; g_A^{\frac{1+\chi}{1-\alpha}} \; g_K^{1+\chi} \; g_F^{-(1+\chi)}.$$

Solving for g_K, we obtain

$$g_K = g_H^{-\frac{1}{\sigma+\chi}} \; g_A^{-\frac{\alpha(1-\sigma)+\sigma+\chi}{(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{1+\chi}{\sigma+\chi}}, \tag{20}$$

which coincides with expression (4.11) in Solis-Garcia and Xie (2018). Plugging (20) into (19), after simplification we get

$$g_N = g_H^{-\frac{1}{\sigma+\chi}} \; g_A^{-\frac{\alpha(1-\sigma)}{(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{1-\sigma}{\sigma+\chi}}, \tag{21}$$

which coincides with expression (4.12) in Solis-Garcia and Xie (2018).

Solving for g_S in (18) we have

$$g_S = g_A^{\frac{1}{\eta}} \; g_K^{\frac{1}{\eta}} \; g_N^{-1}. \tag{22}$$

Plugging the expression (20) for g_K and the expression (21) for g_N into the former equation (22), we obtain

$$\begin{aligned} g_S &= g_A^{\frac{1}{\eta}} \left(g_H^{-\frac{1}{\sigma+\chi}} \; g_A^{-\frac{\alpha(1-\sigma)+\sigma+\chi}{(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{1+\chi}{\sigma+\chi}} \right)^{\frac{1}{\eta}} \left(g_H^{-\frac{1}{\sigma+\chi}} \; g_A^{-\frac{\alpha(1-\sigma)}{(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{1-\sigma}{\sigma+\chi}} \right)^{-1} \\ &= g_A^{\frac{1}{\eta}} \; g_H^{-\frac{1}{\eta(\sigma+\chi)}} \; g_A^{-\frac{\alpha(1-\sigma)+\sigma+\chi}{\eta(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{1+\chi}{\eta(\sigma+\chi)}} \; g_H^{\frac{1}{\sigma+\chi}} \; g_A^{\frac{\alpha(1-\sigma)}{(1-\alpha)(\sigma+\chi)}} \; g_F^{-\frac{1-\sigma}{\sigma+\chi}} \\ &= g_H^{-\frac{1-\eta}{(\sigma+\chi)\eta}} \; g_A^{-\frac{\alpha[(1+\chi)-\eta(1-\sigma)]}{\eta(1-\alpha)(\sigma+\chi)}} \; g_F^{\frac{(1+\chi)-(1-\sigma)\eta}{(\sigma+\chi)\eta}}. \end{aligned}$$

Thus, we get the true expression (13) for g_S, instead of the wrong expression (4.14) reported in Solis-Garcia and Xie (2018). The error in Solis-Garcia and Xie

(2018) seems to come from inserting (20) and (21) into the erroneous formula $g_S = g_A^{1/\eta} g_K^{1/\eta} g_N$ rather than using the correct equation (22), $g_S = g_A^{1/\eta} g_K^{1/\eta} g_N^{-1}$.

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