Reply to “The Limitations of Growth-Optimal Approaches to Decision Making Under Uncertainty”

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We are writing in response to Ford and Kay (2023), where the authors criticise a decision theory which emerges in the field of ergodicity economics (EE), contrasting it with another from expected-utility theory (EUT). We will refer to the two models as EE and EUT, but we emphasise that both ergodicity economics...
and expected-utility theory are labels for fields much broader than what we discuss here. We believe that the criticism of the EE model is due to a misunderstanding on the part of the authors of Ford and Kay (2023), and we thank them for bringing this to our attention so that it may be clarified.

Instead of providing a point-by-point reply, we limit our response to two points, which we hope will unlock the key misunderstanding and clarify where other apparent disagreements come from.

First, we feel the authors have missed an important point about the relationship between EE and EUT. A mapping between the models exists, but the key condition which needs to be satisfied for this mapping to hold is that the utility function in EUT is chosen to be the ergodicity transformation of EE. It seems that the authors believe that the key condition is merely sufficiently long time scales, but this is not the case. We therefore clarify the relationship between the two fields by specifying exactly the mapping between EE and EUT.

Second, we believe we have identified a misunderstanding regarding convergent and non-convergent properties of random walks, which may have led to the first misunderstanding above as it led the authors to write that “final wealth will almost always be what the time average predicts” (Ford and Kay 2023, 317). In the stochastic processes typically studied in EE, the appropriately defined growth rate converges in the long run with probability one to its time average (and expected value). However, this does not imply, as the authors write incorrectly, that final wealth converges to its time average. Clarifying this misunderstanding will help resolve their concerns about other aspects of EE. We provide exact computations for the Peters coin toss discussed by Ford and Kay.

**Formal setup**

Both the EE model and the EUT model make use of a variable which represents the wealth of a decision-making agent. However, the way wealth is modelled is different in the two cases, and consequently, so is the way decision-making is modelled. A mapping between the models exists; that is, we can specify conditions under which they are equivalent. Generally, they are not equivalent, and it is important to state the exact conditions for the mapping to hold in order to specify the relationship between the models.

**EE model**

The formal setup for the EE model is illustrated in Figure 1. It is a choice between two stochastic processes, \( x_A(t) \) and \( x_B(t) \) (Peters and Adamou 2018; Carr...
and Cherubini 2020), representing wealth over time. The processes are chosen so that there exists a monotonically increasing transformation, \( f(x) \), whose increments, \( \delta f(t) = f(x(t + \delta t)) - f(x(t)) \), are ergodic. In particular, the time average of these increments is identical to the expected value of the increments,

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t} \delta f(t + \tau \delta t) = \mathbb{E}[\delta f]. \tag{1}
\]

**Figure 1.** Left: EE operates on stochastic processes, \( x_A(t) \) and \( x_B(t) \) (blue random lines, here geometric Brownian motions). It applies an ergodicity transformation \( f \) (here the logarithm) which produces \( f(x_A(t)) \) and \( f(x_B(t)) \) (red random lines, Brownian motions). These transformed processes are linear in time. Their slopes \( \frac{\delta f}{\delta t} \) converge to the time-average growth rates (slopes of the straight red lines) as \( \delta t \) becomes large. Right: increments \( \delta f \) over a single time unit (red lines, limited to the first 500 time units for clarity). These have the ergodic property that their expected value equals their time average. The increments in the original processes, \( \delta x \) (blue lines), do not have this property, are unstable and not suitable for many computations of interest.

Because of this mean-ergodicity property, the transformation \( f \) is called the ergodicity transformation. The rate of change of the ergodic increments is the appropriately defined growth rate for the process,

\[
g = \frac{\delta f}{\delta t}. \tag{2}
\]

**EE decision axiom:** According to the EE model, agents choose the process which maximizes the time-average or, equivalently (because of the ergodic property), the expected value of the growth rate \( g \).

**Motivation:** Agents in this model can be thought of as representations of people who make decisions in a financial context, where \( x \) represents wealth. By maximizing the time average of equation (2), agents maximize the long-term.
growth rate of their wealth. In the long run, agents who act in this way become wealthier than agents who act differently.

**EUT model**

The formal setup for the EUT model is different; see Figure 2. Here, we deal with a choice between two random variables (not between two stochastic processes), $y_A$ and $y_B$, representing wealth. A monotonically increasing transformation $u(y)$, called the utility function, is defined.

_EUT decision axiom:_ According to the EUT model, agents choose the random variable which maximizes the expected value of the utility function, $\mathbb{E}[u(y)]$.

*Motivation:* Agents in this model can also be thought of as people who make decisions in a financial context, where $y$ represents wealth at a future point in time. The utility function may be thought of as a quantification of how a given level of wealth relates to its subjective value. A concave utility function, for instance, represents a person who assigns less value to an extra dollar as wealth increases. By using the expected value of utility as their maximand, agents weight utility according to the probability of attaining it. Because the utility function can be freely chosen, this model can describe many different behaviours. In EE, both wealth and utility are maximized as time passes. There is no similar physical motivation for EUT; see the next section.

**Mapping EE and EUT**

A stochastic process is a family of random variables parameterized by time. We can, therefore, move from the setup of EE to the setup of EUT by specifying the current time, $t$, at which we wish to evaluate wealth under the processes $x_A$.
and $x_B$ as it will have evolved by the later time, $t + \delta t$, see Figure 3. To establish the mapping, we identify the random variables thus derived from the stochastic processes as the random variables required for an EUT treatment,

$$x_A(t + \delta t) = y_A \text{ and } x_B(t + \delta t) = y_B.$$ \hfill (3)

Under the EE model, we compute the time-average growth rates of wealth under $A$ and $B$ as the rate of change in the expected value of ergodicity-transformed wealth,

$$\bar{g} = \frac{1}{\delta t} \{ E[f[x(t + \delta t)]] - f[x(t)] \}.$$ \hfill (4)

The factor $\frac{1}{\delta t}$ does not affect the ranking of the time-average growth rates for $A$ and $B$, and we can drop it from equation (4). Further, because current wealth is identical for both processes $A$ and $B$, subtracting $f[x(t)]$ in equation (4) does not affect the ranking either, and we can also drop it. Maximizing equation (4) is therefore equivalent to maximizing $E[f[x(t + \delta t)]]$.

Completing the mapping, we summarize that maximizing equation (4) is equivalent to maximizing expected utility in the special case where

- future-wealth random variables $x_A(t + \delta t)$ and $x_B(t + \delta t)$ considered in the EE model are the random variables $y_A$ and $y_B$ considered in the EUT model.
- the ergodicity transformation considered under EE is the utility function considered under EUT.

**Relationship between EE and EUT**

The EE model and the EUT model are equivalent under the restrictive conditions specified in the previous section. It seems interesting to us to highlight what differences emerge when only the second condition is violated, that is, when...
the utility function is not the ergodicity transformation, \( f \neq u \). In this case,

- Only agents who act according to EE and maximize the time-average growth rate of wealth, \( \bar{g} \), also maximize utility, \( u \), as time passes.
- Agents acting according to EUT and maximizing expected utility, \( E(u(y)) \), do not maximize utility, \( u \), as time passes.

These points are rarely stated but they constitute an important limitation of EUT. EUT places great emphasis on defining utility as its quantification of subjective value. In particular, EUT holds that it is better to use utility than money when attempting such a quantification. One might expect the formalism of EUT to guarantee that utility itself—the object of desire by definition—would be maximized over time by agents acting according to its behavioral criterion. But this is not the case.

In contrast, EE focuses on maximising the time-average (or expected) growth rate of wealth. In doing so, it guarantees that EE agents, unlike EUT agents, maximize not only wealth but also utility as time passes. This follows from the assumption that utility is monotonically increasing in wealth (see section “Formal setup,” subsection “EUT model” above). As time passes, EE agents are guaranteed to do better than EUT agents, in terms of wealth and utility.

That EUT maximizes expected utility but does not maximize utility over time is a direct consequence of the non-ergodicity of the wealth process and the utility process it induces. Ergodicity implies that expected value and time average are identical, and therefore in an ergodic utility process it is guaranteed that optimizing the expected value of utility also optimizes the time average of utility. However, the processes usually considered in EE are not ergodic. These include standard models in finance and economics, for example Brownian or geometric Brownian motion. Here, the equality of expected value and time average does not hold, and consequently expected-utility theory does not optimize utility over time.

We feel that we should clarify another comment by Ford and Kay because it allows us to highlight the astonishing experimental results obtained by the Copenhagen group. EE, as Ford and Kay put it, violates the axiom of completeness and leads to “inconsistent decision making” (2023, 316). This is a strange way of saying that for a given pair of random variables \( y_A \) and \( y_B \), EE can conclude that either \( A \) or \( B \) is preferable if the dynamic is left unspecified. Of course the same is true of EUT, if the utility function is left unspecified. It is unclear to us what practical problem arises from this, but this served the Copenhagen group in their attempts to put EE to the test.

Recall that the ergodicity transformation is given by the dynamic of the
stochastic process $x(t)$. It is a subtle but important detail that two different stochastic processes, both starting at $x(t)$, can yield identical distributions at a later time $x(t + \delta t)$. In particular, the processes may have different ergodicity transformations. Therefore, an EE agent evaluating what looks to EUT as the same situation, namely the same random variables $y_A = x_A(t + \delta t)$ and $y_B = x_B(t + \delta t)$, can arrive at different preferences depending on the process which generates the random variables.

From the perspective of EUT, because it does not take dynamic information into account, one could call this a violation of the completeness axiom of EUT. Put more prosaically, this is a case of a hidden variable, and once the dynamic is specified, the preferences of EE agents satisfy completeness. The case of the EE agent using different dynamics is equivalent to the case of an EUT agent using different utility functions. Of course, using different utility functions, the EUT agent can also arrive at different preferences, violating completeness in the same sense. But once a utility function is specified, also EUT preferences satisfy completeness.

However, the fact that EE preferences change according to dynamics enables experimental explorations of the theory. By manipulating the wealth process, $x(t)$, in simple gambling tasks in a laboratory setting, experimenters can control the ergodicity transformation. This makes it possible to test whether real human subjects behave according to idiosyncratic utility functions or according to circumstantial ergodicity transformations. To the great astonishment of most of us, the latter is often the case: fitting the EUT model to choices made under different dynamics reveals that people change their apparent utility functions to coincide with the relevant ergodicity transformation (Meder et al. 2021; Skjold et al. 2023).

By its construction, EUT does optimize expected utility, but because the wealth process is not ergodic, this object is different from what materializes for the decision-making agent. While the expected value is approximated by the average over large statistical samples, it is not generally a quantity of interest for an individual decision maker.

We illustrate this with the Peters coin toss. An agent is offered a repeated fair coin toss, where heads leads to a 50 percent rise in wealth and tails leads to a 40 percent drop. The coin toss is also discussed in Ford and Kay (2023), a video about it is available via ergodicity.tv (link), and an interactive blog post is available on ergodicityeconomics.com (link). It is a special case of the multiplicative binomial process (Redner 1990).

First, we evaluate the gamble on offer using the EE model. Here, two stochastic processes are compared. The first stochastic process is trivial: if the agent rejects the gamble, wealth will be unchanged at its current level, $x_A(t + \delta t) = x(t)$.

The second stochastic process arises from the agent accepting the gamble,
and \( x_B(t) \) is a random walk in logarithmic space. The ergodicity transformation for this particular dynamic is the logarithm,

\[
f(x) = \ln(x),
\]

(5)

and both \( \delta \ln (x_A) \) and \( \delta \ln (x_B) \) are ergodic. This means the appropriately defined growth rate is

\[
g = \frac{1}{\delta t} \ln x(t + \delta t) - \ln x(t)
\]

(6)

and its time average is

\[
\bar{g} = \frac{1}{\delta t} \mathbb{E}[\delta \ln x].
\]

(7)

To be explicit: it is the ergodic property of \( \delta \ln x \), which allows us to maximize the time average of \( \delta \ln x \) (and thereby \( \bar{g} \)) by maximizing the expected value \( \mathbb{E}[\delta \ln x] \). This maximisation guarantees that we end up with greater wealth (and utility) in the long-time limit. Evaluating for both processes, we find \( \bar{g}(x_A) = 0 \) per round and \( \bar{g}(x_B) \approx -0.05 \) per round. The EE agent picks the process with the greater time-average growth rate, rejects the gamble and remains at \( x(t + \delta t) = x(t) \).

Second, we evaluate the gamble on offer using the EUT model. To be able to do this, the agent needs to specify its utility function. To illustrate the problem with a simple example, let’s say the agent has linear utility, \( u(y) = y \), although the situation we’re about to highlight also arises with many other utility functions.

For simplicity, we let the agent evaluate utility after one round, although nothing changes if the agent were to evaluate utility after an arbitrary number of rounds. Here, two random variables are compared. The first random variable is trivial, namely, wealth remains unchanged if the gamble is rejected, and \( y_A = x(0) \). The expected utility associated with this random variable is, trivially,

\[
\mathbb{E}[u(y_A)] = x(0).
\]

(8)

The second random variable, \( y_B \), takes the value \( 1.5x(0) \) with probability 1/2 and \( 0.6x(0) \) with probability 1/2. The expected utility associated with \( y_B \) is
\[ \mathbb{E}[u(y_B)] = \frac{1}{2}[u(1.5x(0)) + u(0.6x(0))] = 1.05x(0). \]  

Since \( \mathbb{E}[u(y_B)] \) is greater than \( \mathbb{E}[u(y_A)] \), the EUT agent with linear utility will always choose to participate in the gamble.

However, as we’ve seen in the EE analysis, the time-average growth rate of wealth is negative for this gamble: the probability that the agent loses money approaches 1 over time. Because the agent’s utility function is monotonically increasing, losing money means losing utility. This illustrates that the EE agent maximizes utility over time, whereas the EUT agent only maximizes expected utility but not actual utility. In the simple example we’ve given, the EUT agent loses utility as time passes, whereas the EE agent does not. Ergodicity in multiplicative dynamics is broken in such a way that the expected value of many monotonically increasing utility functions does not indicate how utility actually behaves with probability 1 over time. In many cases, as in our example, an increasing expected utility \( \mathbb{E}[u(t)] \) is accompanied by systematically decreasing actual utility, \( u(t) \).

**Wealth uncertainty diverges while growth-rate uncertainty vanishes**

That the EE model produces such different outcomes from the EUT model is a profound consequence of uncertainty, which we believe was overlooked in Ford and Kay (2023, 317), where the authors write: “The time averages used in [the EE model] correspond to a situation where there is no measurable uncertainty—final wealth will almost always be what the time average predicts.”

This sentence seems to us to reflect a misunderstanding. The EE model uses the time average of the growth rate of wealth, \( \bar{g} \), as its decision criterion. It does this because this quantity converges to a meaningful finite value, and such a simple scalar is needed to rank the stochastic processes \( x_A \) and \( x_B \). However, this convergence does not imply that wealth itself, \( x(t) \), converges to a value predicted by the time-average growth rate. For instance, in the Peters coin toss, if we average the growth rate over a finite time \( T \), then its variance vanishes as \( 1/T \). The variance of wealth, on the other hand, diverges exponentially. Contrary to Ford and Kay’s statement, there is great uncertainty in final wealth, in the sense that it diverges in the limit \( T \to \infty \), whether we measure it by variance or other relevant ways of measuring uncertainty (see Appendix). Specifically, in the Peters coin toss with...
linear utility, the uncertainty leads to the ordering under EUT being different from the ordering under EE, whether after a finite or divergent number of rounds. This means that the uncertainty in terminal wealth is not only measurable but crucially important in the case under consideration.

We can only speculate here, but this misunderstanding may explain other comments by Ford and Kay (2023), which we find difficult to understand otherwise. For instance, the authors write:

- “[O]ne might expect that, as the gamble’s length goes to infinity, the predictions of EUT would approximate the growth-optimal predictions” (Ford and Kay 2023, 318).
- “[U]tility approaches [the EUT model] and growth-optimal approaches [the EE model] are likely to give the same answer in many cases” (ibid., 326).
- [The difference between the growth rate of the expected value and the time-average growth rate] “is mechanically incorporated in an EUT analysis of choices as only final outcomes are considered” (ibid., 318).

As we say in the “Mapping EE and EUT” section above, the two approaches only give the same answer if the utility function, $u$, is chosen to be the ergodicity transformation, $f$. However, the authors seem to be under the impression that the EE model essentially considers wealth far in the future and that this wealth is known with “no measurable uncertainty” (Ford and Kay 2023, 317). If this were the case, then their statement would be true: EE would compare known wealths $x_A(t)$ and $x_B(t)$ at some large $t$, and EUT would compare utilities $u[x_A(t)]$ and $u[x_B(t)]$ (we would be allowed to replace $E[u[x_A(t)]]$ by $u[x_A(t)]$ when there’s “no measurable uncertainty” in $x_A(t)$). Because $u(x)$ is assumed to be a monotonically increasing function, the preference orderings of A and B would be the same under both models. If this were true, one would presumably use neither EE nor EUT and just compare asymptotic wealth or, equivalently, utility. But none of this is actually the case, and uncertainty in $x(t)$ grows beyond all bounds with $t$.

Wealth, in absolute terms, in the Peters coin toss goes to zero with probability one. In this sense, asymptotic wealth is known in this particular case. However, this is an asymptotic statement which must be interpreted carefully. The statement that terminal wealth is known with “no measurable uncertainty” is not correct, even in this special case as should be clear from the fact that expected wealth diverges while most probable wealth goes to zero. Uncertainty in terminal wealth diverges with time if we measure it by standard deviation; it also diverges if we use relative measures of uncertainty (see Appendix); most significantly, in the
present case (coin toss with linear utility), the uncertainty in terminal wealth leads to expected utility of terminal wealth being positively divergent, whereas utility of the terminal wealth which is approached with probability one is zero. This illustrates once more that EE is a very different model than EUT.

Experiments and the role of psychology

Ford and Kay are critical of experimental work carried out to establish the realm of validity of the EE model (Meder et al. 2021; Vanhoeyweghen et al. 2022). We agree with many of the criticisms and will provide a detailed response in a separate reply. We note that experimental design is always subject to constraints, such as ethical and financial considerations. Nor can a laboratory experiment ever be truly realistic, and in the process of designing, choices must be made. We have actively sought critiques of the designs of the existing experiments and have ourselves spent a great deal of time discussing weaknesses and alternatives. We have taken seriously all critiques we have received and incorporated them into the next phase of experiments (Skjold et al. 2023).

Controlled laboratory experiments are of interest to psychologists and neuroscientists. The experiments help clarify in how far EE provides a new behavioral baseline, deviations from which may yield insights into individual psychology. Ford and Kay’s concern that EE “claims to provide an objective justification for decision making without the need to refer to individual psychology” (2023, 315) is therefore misplaced. It is not by accident that individual psychology (represented by idiosyncratic utility functions in EUT) is excluded from the EE model. Neuroscientifically, it is extremely exciting that behavior which used to be thought of as idiosyncratic, trait-like, and changeable only on evolutionary time scales can actually be altered by a simple intervention in the experimental environment on time scales of hours rather than millennia.

Conclusion

We conclude with a quote from the memoir of Giorgio Parisi. Referencing the difficulty of transmitting ideas across traditional disciplinary boundaries, he wrote: “I believe that with a lot of good faith and a lot of patience, at least in the majority of cases, it is possible to arrive at shared conclusions. Or at the very least to clarify where our disagreements come from” (Parisi 2023, 18). We hope that this will be the case in the present context too.
Appendix

We provide a technical analysis of the coin toss game in an appendix available at the journal’s website (link).

Data and code

All data shown here are for illustration purposes only and were randomly generated. Codes to reproduce all data and figures are available from the journal’s website (link) and are archived at Zenodo (link).

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