The Limitations of Growth-Optimal Approaches to Decision Making Under Uncertainty

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A fundamental problem in the analysis of choice under uncertainty is that it is not obvious what constitutes optimal behaviour for an individual. Within economics assumptions have traditionally been made to justify various approaches: for example Expected Utility Theory (EUT) is justified by John von Neumann and Oskar Morgenstern’s axioms about the preferences individuals have over lotteries. Weakening the axioms required by a model of decision making has clear attractions, and optimal growth theory represents a notable attempt to do so. John Kelly (1956, 925) showed that when returns compound, there is an asymptotically-optimal fraction of your wealth to offer in any given round of a repeated gamble, and that a gambler following this method “maximizes the expected value of the logarithm of his capital.” This analysis has subsequently been extended to portfolio theory, in particular by Henry Latané. It has long been recognised, however, that this approach in general conflicts with EUT and requires different assumptions about what decision makers are aiming to achieve.

More recently, Ole Peters, who is a physicist, and collaborators, including the late Murray Gell-Mann, have drawn on ergodic theory to argue that optimal growth provides a general foundation for theories of decision making in economics (Peters

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‘Ergodicity economics’ (EE) focuses on time averages as an alternative to the use of the expectation operator in mainstream economic theories of decision making under uncertainty, in particular EUT. EE transforms accumulation dynamics into ergodic processes (for which the expected value and time average coincide). EE then bases its predictions on the limit of the time average of the growth rate—as with other growth-optimal theories, the justification is that, in the limit, the process with the highest time-average growth rate will result in the most wealth (see Peters and Gell-Mann 2016, eqs. 2 and 5). EE is therefore distinct from other growth-optimal approaches for two reasons: it claims to provide an objective justification for decision making without the need to refer to individual psychology; and its proponents claim that it makes more metaphysical sense than alternative models of decision making.

This work on decision theory, and the associated criticisms of utility theories such as EUT, form the core of the ‘ergodicity economics’ research programme. The claims are forceful: readers are told that EE “resolve[s] the fundamental problem of decision theory, therefore game theory, and asset pricing” (Peters and Gell-Mann 2016, 8). We show in this critique that the model EE proposes is unsatisfactory in several respects and does not represent an improvement on utility theories. Furthermore, we show that their criticisms of utility theories are misguided and based on a misunderstanding of how they work. In particular, many of the distinctions drawn between utility theories and EE arise because, when analysing the same problem, Peters and Gell-Mann use different information sets, as well as different models. We argue that there is no justification for using different information sets and that EUT looks much more credible when it is given informational parity. As a result, we reject their claim to have resolved any fundamental problem, and we are sceptical of EE’s value for economists.

Most of the other work in the ‘ergodicity economics’ programme is based on the same fundamental approach and is therefore vulnerable to our critique. For example Peters (2011) claims to solve the St. Petersburg Paradox by altering the problem so that the individual faces multiple opportunities to participate in the gamble. His solution rests on the individual maximising the growth rate of their wealth and does not address what seems to us to be the core issue, which is that, in a world with truly unbounded potential payouts in terms of subjective satisfaction, it is unclear that the ‘paradoxical’ intuition we display—that we would not, in the real world, pay very much for St. Petersburg-type gambles—applies. It is therefore not a convincing demonstration of the limitations of EUT, or of the merits of

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3. A “dynamic” is specified by the nature of the interaction of the random variable with the state variable, i.e., wealth. For example if the realisation of the random variable is added to wealth the dynamic is additive; if wealth is multiplied by the realisation it is multiplicative, etc.
Another paper on insurance, co-written by Peters with Alexander Adamou (2017), similarly assumes that individuals wish to maximise their growth rate and contrasts this to an analysis based on risk aversion. They neglect to include the same information set in their EUT calculations (i.e., that individuals know that they will face similar situations repeatedly) and so mischaracterise that approach.

It is worth noting that ‘ergodicity economics’ has also been applied to questions about inequality, for example by Adamou and Peters (2016). This work is, conceptually, entirely distinct from the work on decision theory (although the mathematical background is similar), and it is not susceptible to the criticisms presented here. Danny Dorling (2016) has argued persuasively that the model of inequality they propose does not explain many of the data’s most interesting features, but this is an empirical question and it is possible that reformulations of the model may be more successful.

The key insight of optimal growth approaches is also their central flaw: they posit a theory of decision making under uncertainty without uncertainty. Paul Samuelson (1971) showed that for a certain class of utility functions the limit was an inappropriate guide for any finite process, but this limitation was also implicitly mentioned by Kelly (1956, 920): “It is surely true that if the game were to be stopped after N bets the answer to this question would depend on the relative values (to the gambler) of being broke or possessing a fortune.” However the situation is worse than this: optimal growth approaches necessarily violate EUT, in particular the axiom of completeness. In other words, they lead to inconsistent decision making.

**Terminology**

The terminology used by different authors varies. Von Neumann and Morgenstern (1955) refer to a random drawing of one of a set of payoffs, weighted by a defined probability distribution, as a “lottery.” When several lotteries are combined—for example if the payoff of a lottery is participation in another lottery—the overall structure is referred to as a “compound lottery.” In contrast, Peters (2019) refers to “gambles,” which are series of random drawings applied in sequence to a transformed variable, typically wealth.

Throughout this paper we will refer to individual drawings as lotteries and a series of lotteries, each applied to the wealth resulting from the previous lottery, as a gamble: $G$. Gambles therefore have a length $t$. A gamble of length 1 is just

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4. Note that lotteries are over overall outcomes, so a coin toss to gain or lose $100 is a different lottery if you have a starting wealth of $1,000 or $10,000.
a lottery. For concision we refer to approaches that result in optimal growth in the limit as ‘growth optimal.’ We note here, however, that for the finite cases we are considering this is a misnomer, as the growth rate is a random variable and a ‘growth optimal’ approach will not always lead to higher growth.

By ‘psychology,’ we mean the subjective valuation of objective experience: for example, different people will value different consumption bundles differently, leading them to make different choices.

**Optimal growth**

For any gamble where returns compound total growth $G$ after $t$ draws is given by:

\[ w_t = w_0 \cdot \prod_{i=1}^{t} g_i = G = \prod_{i=1}^{t} g_i \]  

(1)

where $g_i$ is the realisation of the $i$th lottery in the gamble. We can rearrange this to give the per-stage growth rate (a time average) $\tau_t$:

\[ \tau_t = \frac{G^t}{(\prod_{i=1}^{t} g_i)^{\frac{1}{t}}} \]  

(2)

If we then take the natural log of both sides, we have:

\[ \ln \left( \tau_t \right) = \frac{1}{t} \sum_{i=1}^{t} \ln \left( g_i \right) \]  

(3)

This is the sample average of the gamble transformed by the natural logarithm. If the sequence of growth rates $g_i$ is stationary and ergodic and $\mathbb{E}[\ln(g_i)]$ is finite then by the Ergodic Theorem, as $t \to \infty$, $\ln(\tau_t) \to \phi$ almost surely where $\phi$ is some constant. From this it follows that $\tau_t \to e^\phi$ almost surely. The growth rate will therefore be $e^\phi$ with probability 1.

The time averages used in growth-optimal models correspond to a situation where there is no measurable uncertainty—final wealth will almost always be what the time average predicts. Decisions reduce to choosing the gamble which gives the highest wealth with (effective) certainty. But we do not live in the limit—to paraphrase only slightly the famous dictum of John Maynard Keynes, in the limit we are all dead.5 Economic agents necessarily have finite lives, and may also be

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5. Keynes’ original statement was “In the long run we are all dead” (Keynes 1924, 80).
interested in outcomes over shorter segments of that, as well as averages over the whole of these lives. What insight does growth optimality offer in these cases?

Peters (2019, eq. 2) describes a lottery (hereafter referred to as ‘Peters’ bet’) with an equal chance of increasing your wealth by 50 percent or decreasing it by 40 percent. If this is applied repeatedly over time to total wealth, it is very likely to leave one with almost nothing—the time-average growth rate is negative—even though the expected gain in each round is positive. If this seems counterintuitive, suppose you start with $100 and play the bet for two rounds, and win the first round and lose the second. Winning the first round increases your wealth to $150, but losing the subsequent round decreases it to $90—less than you started with. However if we take the average of the four equally likely final outcomes—$36, $90, $90, $225—it is $110.25. This ‘volatility tax’ is a well-known phenomenon in financial markets and is mechanically incorporated in an EUT analysis of choices as only final outcomes are considered (assuming, as with the simple growth-optimal models discussed here, there is no discounting and no consumption throughout the gamble). However one might expect that, as the gamble’s length goes to infinity, the predictions of EUT would approximate the growth-optimal predictions.

Samuelson (1971)’s contribution is to show that this is not the case for many utility functions. Noting that for any finite gamble the outcome is a random variable, rather than a deterministic number, he shows that an agent with a certain type of utility function—for example a risk-neutral agent described by \( u(w) = w \)—would want to take Peters’ bet regardless of its length (and regardless of whether or not they could decide to stop gambling after each stage). Although a risk-neutral agent taking the gamble will probably end up with almost nothing, for them the chance of a very large payoff will always outweigh that risk. In the limit case those high-value outcomes are suppressed, explaining why EUT might appear to coincide with growth optimality.

**Comparison to expected utility theory**

We have seen that growth-optimal models can be justified by a riskless ‘limit’ world, and in such a world preferences over spreads of possible outcomes are unnecessary: the only preference ordering necessary is preferring more to less. In contrast, EUT can incorporate many preferences which give rise to risk aversion (in the sense that the more we have of something already the less value we tend to get from additional units—winning $1M after going bankrupt makes more of a difference than winning $1M when one has just become a millionaire). Yet risk-neutral EUT is not growth-optimal. So which preferences over risk do growth-optimal models assume?
Choosing a gamble to maximise the growth rate of wealth looks equivalent to choosing a specific utility function: if the dynamics are multiplicative we maximise the expected value of \( u(w) = \ln(w) \); if they are additive we maximise the expected value of \( u(w) = w \). Peter Carr and Umberto Cherubini (2020) draw on this insight and vary a stochastic clock to show that a variety of utility functions can be justified in this way, and Peters and Adamou (2021) sets out a general way to find a correspondence between utility functions and dynamics (where this exists). However these ‘utility functions’ are artefacts of the transformation made to find the optimal growth rate of the gamble and depend on the dynamics of wealth accumulation: they are not representations of preferences over lotteries.

Given this, Samuelson (1971) can be extended to show that growth-optimal behaviour necessarily violates the axioms of EUT because it is incompatible with any utility function. Any non-linear utility function is incompatible with taking growth-optimal additive gambles: a strictly concave region means that, in that region, an individual would not take an additive gamble with a very slightly positive time average; and in a strictly convex region they would take a gamble with a very slightly negative time average. However a linear utility function means that an individual would accept Peters’ bet, which we have seen EE rejects.

A consequence of growth-optimal approaches contradicting EUT is that they necessarily violate at least one of von Neumann and Morgenstern’s axioms. The axiom violated is completeness: they fail to represent (or, in more normative terms, establish) a preference ordering over many gambles. This occurs in three distinct ways.

Firstly, they are in general unable to rank gambles with different dynamics against each other. For example, how are we to decide between an additive gamble with a time average growth of 5 units per round and a multiplicative gamble with a time average growth of 1 percent of wealth per round?

Secondly, all gambles of length 1 can be described with both additive and multiplicative dynamics, potentially giving different preference orderings compared to a ‘gamble’ which leaves wealth unchanged with certainty. For example Peters’ bet, when expressed as an additive gamble, is preferred to this static ‘gamble’; but when expressed as a multiplicative gamble it is not. But when both gambles are of length 1, they are exactly the same. Therefore some gambles must be at least some minimum length—which might vary depending on the gamble—to be comparable.

Finally, some time-varying gambles are not comparable. To show this, sup-

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6. It is worth noting that Samuelson (1963, 110–111) shows that a system based on choosing the option most likely to leave you better off results in a violation of transitivity, but this is not necessarily the strategy used by EE.
pose every type of gamble has some minimum length \( t_{\min}(G) \) such that, when it is
of length \( t \geq t_{\min}(G) \) it is comparable to all other gambles of sufficient length. As we
saw above, for at least some gambles this must be greater than 1, and we will see
later that we might want it to be very large.

Figure 1. Three deterministic gambles

Now consider three ‘deterministic’ gambles, i.e., ones where the change in
per-period growth is certain, with multiplicative dynamics, shown in Figure 1.

The red gamble \( G_c \) grows at a constant rate of \( \frac{g}{\alpha} \) per period. The blue gamble
\( G_+ \) has a growth rate \( \tau_+(t) \) which is increasing in \( t \) and tends towards \( \alpha g \) where \( \alpha > 1 \).
It is defined recursively:

\[
\frac{d_t}{w_t} = \frac{d_{t-1}}{w_{t-1}} \left( \frac{t}{t+1} \right)
\]

which gives the general form

\[
\frac{d_t}{w_t} = \frac{d_0(\alpha g)\left( \frac{t}{t+1} \right)}{(t+1)!(t+1)}
\]

The green gamble \( G_- \) has a growth rate \( \tau_-(t) \) which is decreasing in \( t \) and tends
towards 0. It is defined recursively:
\[ w_t = w_{t-1} \left( \frac{t+1}{t} \right) \]  

which gives the general form

\[ w_t = w_0 \left( \frac{(t+1)!}{t!} \right) \]

Taking the time averages of these gambles, EE gives the ordering \( G_+ \succ G_C \succ G_- \). However as Figure 2 shows for some values of \( t \), \( w(G_+(t)) < w(G_C(t)) < w(G_-(t)) \). A natural conclusion is that \( t_{\text{min}} \) for time-varying gambles must be sufficiently high such that there is no constant deterministic gamble \( G'_C \) such that \( G'_C \prec G_+ \) where \( w(G'_C(t_{\text{min}})) > w(G_+(t_{\text{min}})) \). Otherwise it would be the case that growth-optimal models suggest that people prefer certainly receiving less money to certainly receiving more. However it will always be possible to create such a gamble by setting its growth rate marginally below \( \alpha_g \). Therefore there is no possible value of \( t_{\text{min}} \) at or above which this gamble can be ranked, and so this type of gamble also cannot be compared by these models. As well as revealing a further violation of completeness, this demonstrates why using a limit result for finite problems can be—and in this case is—inappropriate: merely examining the time average growth rate results in a clear ordering with no warning that there is no finite length for which this ordering makes sense.

### Psychology is fundamental to decision making

The contradiction between the predictions of EUT and growth-optimal models reveals that making psychological assumptions is fundamental to theories of decision making under uncertainty: when dealing with uncertain outcomes there cannot be a simple, generally applicable, and completely persuasive rule for what all decision makers should choose. As a result, claims that these approaches are valuable because they do not “appeal to an intangible psychology” (Peters 2019, 1218) become arguments against them.

Carr and Cherubini (2020, 3) run up against the same problem: they explicitly attempt “to reconcile Kelly and Samuelson” without appealing to utility. However their findings—interesting though they are—are again based on limit cases which

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7. One way of avoiding this issue would be to make \( t_{\text{min}} \) a function of both gambles being compared, rather than just an individual gamble. The problem with such an approach is that it would inevitably involve examining both entire distributions of potential outcomes at particular points in time and judging whether one was different enough to the other to justify distinguishing between the gambles. At this point the decision criteria is not really about the time average and is very close to EUT.
cannot be reached. In the real world, uncertainty of outcome remains, and different people will have different preferences across these. Samuelson and Kelly remain unreconciled.

We have seen that relying on the limit is insufficient as a justification of growth-optimal approaches. Sophisticated theorists have not attempted to claim that it is: for example Latané (1979, 310) notes that “My interest…is in the asymptotic qualities of $G$ and the measurement of the probability of adverse dominance… It seems to me that this probability of adverse dominance is especially relevant to corporate and other investment decisions and portfolio management where individual subjective utilities of those involved are difficult if not impossible to determine.” As we noted above, more recently a novel justification has been offered by physicists with a background in ergodic theory.

Their claim is that decision theory must consider what actually happens to an individual, not what might happen to them. For this it needs to be an “ergodic observable”, because for such observables the ensemble and time averages coincide (Peters and Gell-Mann 2016, 1). To create this observable we have to understand the dynamics of a gamble over time: “requiring the specification of a dynamic is requiring the admission that we live through time, not in a supervers of parallel worlds with which we can share resources” (ibid., 3). This is contrasted with EUT: “Expected utility theory computes what happens to a loosely specified model of my psychology averaged across a multiverse. But I do not live spread out across a multiverse, let alone harvest the average psychological consequences of the actions of my multiverse clones” (Peters 2019, 1218). EUT therefore stands accused of making two metaphysical errors: ignoring time; and assuming resources can be shared across multiple possible universes. We saw above, however, that it is wrong to say that EUT takes no account of time: time is implicitly incorporated into the potential outcomes explicitly considered, but only the time that will actually elapse during the decision problem. In contrast EE considers all the time that would elapse were the gamble to go on forever. Figure 2 illustrates this difference. (We discuss applying EUT to problems involving time in Appendix A.)

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8. If we have a probability space $(\Omega, \mathcal{F}, P)$ and a measure-preserving transformation $T: \Omega \rightarrow \Omega$, then we can think of each application of the transformation resulting in a new period in time. If the transformation is ergodic then Birkhoff’s Ergodic Theorem states that the time average converges almost surely to the expected value (where it exists). $T$ is ergodic for the measure $P$ iff for any $F \in \mathcal{F}$ such that $T^{-1}(F) \subset F$ either $P(F) = 0$ or $P(F) = 1$: intuitively, if we take a subset of the sample space resulting from the application of the transformation to some other subset, then the former subset will not be contained in the latter, except in trivial cases.
Figure 2. Different perspectives on the same bet, which increases wealth by 50 percent or decreases it by 40 percent, each with equal probability. An EUT analysis takes place at a specific point in time, illustrated here as $t = 5$; an EE analysis includes the infinite unrealised future.

The suggestion that use of the expectation operator invokes multiple worlds is also incorrect. EUT is a *representation* of how people make decisions, not a realistic description. Decision makers do not form an average over outcomes because they think some physical equivalent of the averaging will occur; they can be represented as doing so purely because, if they are described by the axioms in von Neumann and Morgenstern (1955), their choices based on their preferences admit such a representation. In fact EE also uses the expectation operator following an appropriate transformation: this does not mean it is “assuming resources can be shared across multiple possible universes”; just that the expected value happens to be mathematically equivalent to the object it is interested in. Just like in EUT.

It is unclear why, in principle, we would ever want to use any form of summary statistic of the potential outcomes of a decision rather than the entire distribution of outcomes, except to make the problem more tractable: this applies equally to expected values and time averages. EUT is a special case where, given various axioms are met, the use of the expected value of a function happens to encode sufficient information to identify an agent’s preferred choice. It is not a general claim that expected values tell us all we need to know about decision
making. Furthermore, it is true to say that EUT considers multiple potential outcomes, only one of which will occur—but in that it is no different to EE, and we have seen that for real gambles (which necessarily occur in finite time) this fundamental truth cannot be avoided.

We suspect, however, that the limit is the implicit justification for EE, because—in marked contrast to Latané’s work—EE gives no guidance as to when a gamble is long enough for the theory to apply. The following example demonstrates the problems this causes. Consider two gambles on a realistic time scale: in the first, a fair coin will be tossed every month for the next ten years. Each time it lands heads you receive $2; each time it lands tails you pay $1. The expected gain is $0.5 per month. The second is identical, except that if the coin lands heads you receive $102 and if it lands tails you pay $100. The expected gain per month is $1. As this is a case of additive growth, EE predicts individuals will prefer this second gamble to the first. We predict that they would not, because it is much riskier—in general the sort of people who find such gambles attractive have not survived. Furthermore, the two variables which might change our prediction—if the individual is extremely wealthy, and if they are approximately risk-neutral, perhaps because they are a firm with diversified shareholders—are variables in EUT but not EE.

**The purpose of a decision theory**

One point worth emphasising is the different purpose of these two decision theories. As we discuss in Ford and Kay (2023), EUT and its intellectual descendants are descriptive, not normative, theories: ‘of’ rather than ‘about’ decision making. In practice the distinction between descriptive and normative theories has been elided by many economists. To understand both the limits and benefits of EUT, it is helpful to distinguish the two.

The axioms of EUT and similar theories can be split into two groups: the first allow preferences to be described sensibly; the second restrict those preferences. For example, in EUT the axioms of completeness and transitivity allow for a description of an agent’s preferences, whereas the axioms of continuity and independence restrict them.

The descriptive axioms ensure that all of the objects in the agent’s choice set can be ordered against each other with the preference relation, and that there will, in any subset of this choice set, be at least one most-favoured option. What this means in practice is that, given any set of choices an agent faces, we always know what they want to do (or the collection of choices they are indifferent between but prefer to all the other choices). There is no need for any type of utility function to
tell us what they will choose. But it is important to note that this gives no insight into how or why the agent has this particular preference ordering. In that sense, it is psychologically naïve: it simply asserts that the agent already knows exactly what they want in any situation.

The restrictive axioms may or may not be plausible descriptions of an agent’s preferences, but their purpose is to allow the preference ordering guaranteed by the descriptive axioms to be expressed with a utility function. For example, lexicographic preferences may be complete and transitive but they violate the axiom of continuity: this does not imply anything about the rationality of having those preferences, just that they cannot be represented by a continuous utility function.

EUT, therefore, is a descriptive theory which may or may not be a useful model of decision makers depending on the context of the situation they are placed in. It is not a normative theory unless we think the axioms themselves have normative force, and there is good reason to think that they do not. From this perspective, the theory’s capaciousness is a benefit, not a drawback: it can accommodate many different sets of preferences, allowing it to describe many situations; it is no loss that it does not give precise predictions about what all agents will or should do since it makes no claims to be normative.

It is less clear whether EE is descriptive, normative, or both. Empirical work (discussed in Appendix B) suggests that there is hope that it is a descriptive theory, and one which produces contrasting results with EUT. There are also signs that it is a normative theory, on the grounds that optimising the growth rate is the sensible thing to do: for example Peters (2019, 1218) suggests that the author would make certain choices “because the payments correspond to a higher (additive) growth rate of my wealth” and “it’s the growth rate I would optimize.”

## Conclusion

Theories of decision making under uncertainty can be normative, descriptive, or both. Utility-type approaches are necessarily descriptive if the axioms are met, but have a much weaker claim to being normative: the axioms of EUT have come in for significant criticism, most persuasively by Matthew Rabin (2000). In contrast growth-optimal approaches appear to be normative, although as we have noted different theorists present different normative justifications for them.

Whether growth-optimal approaches are descriptive is a more difficult question. We are not aware of empirical studies examining how closely individuals’ portfolios approximate the growth-optimal benchmark, but EE’s metaphysical claims have led to a resurgence of interest in this general area, and two recent
experimental studies—one by David Meder et al. (2021) and another by Arne Vanhoyweghen et al. (2022)—attempt to distinguish empirically between EUT and EE. Unfortunately the too-frequent assumption that EUT ignores time and therefore can be represented as myopic—only looking at the next bet, even when the dynamics are known—leads to biased estimates in Meder et al. (2021). Vanhoyweghen et al. (2022) unwittingly present a situation where the predictions of EUT and EE are, at least in their formulation, identical! A more detailed analysis of both papers is provided in Appendix B.

We expect that, for real decisions which real people face, utility approaches and growth-optimal approaches are likely to give the same answer in many cases. This is especially true for short gambles where the dynamic is not given by the gamble but instead by what will happen to the person’s wealth over their lifetime: for example if we assume that excess wealth gets put into a diversified portfolio then it is likely to grow multiplicatively. From this perspective, we could say that the dynamic is an aspect of someone’s life, rather than a particular gamble, which would tie them down to a single utility. This remains the case for EUT models over time such as in Samuelson (1969) where consumption also depends on investment performance. This is because, as discussed above, EUT approaches automatically incorporate the growth dynamics of different investments: the only reason they differ is that EUT considers the spread of possible outcomes, and when most of the weight of this final probability distribution is over a very small range, we generally expect this to not be a very significant difference—although, as we have seen, it can be. In contrast, a third approach to investment—single-period models with particular objectives, such as Harry Markowitz (1952)’s pioneering variance-aversion model of portfolio choice—do not directly consider the effects of volatility on investments’ performance over time (although variance aversion will implicitly result in similar behaviour). Here, therefore, we would expect the models’ conclusions to diverge to a greater extent.

The great benefit of growth-optimal approaches to decision making is that they describe an underlying characteristic of gambles, which is true regardless of the psychology of any decision maker who happens to face them. However when individuals are faced with uncertainty, psychology will necessarily play a role: this may express itself in maximising the chances of getting rich; in extreme caution (e.g., minimax); or in other ways. There are many drawbacks to utility approaches, and we have discussed them at length elsewhere (Kay and King 2020). Their great benefit, however, is that they explicitly list the conditions under which they will necessarily be valid. We have shown that if these conditions are met then they necessarily contradict growth-optimal approaches, but this should not be seen as a criticism of the benefits growth-optimal perspectives bring, both scientifically and pragmatically. Instead this result should be seen as emphasising the fundamental
importance of psychological factors layered on top of a scientific understanding of the systems, and as emphasising how we cannot remove individuality from the equation when it comes to decisions made under uncertainty. To improve our understanding of decision making we should study these psychological factors, investigating how people actually make decisions.

Appendix A.
Using expected utility theory over time

In their reply to Peters (2019), Jason Doctor et al. (2020) note that EUT is a “static” theory and that economists use dynamic alternatives over time. However for cases like the ones discussed in this paper, where the only wealth individuals derive any direct benefit from is that which is produced at the end of a gamble, it is straightforward to analyse them with EUT. We noted above that EUT ‘mechanically’ incorporates the process by which potential final outcomes are reached. Indeed, EUT defines gambles like these as ‘compound lotteries’ and asserts that only the final outcomes of the compound gamble matter for the decision problem. If we consider a situation where individuals are offered a gamble of a particular length but can choose to participate or not at each stage the situation is a little more complicated, and we need to solve it recursively: so find the utility of every possible final outcome; then look at the preceding decision nodes and compare the utility of taking the lottery they represent to the utility of keeping that wealth for sure, and take the higher value as that node’s utility; etc.

The cases discussed in this paper are a very restricted subset of the ones we are likely to care about. In practice, investment decisions are not about wealth in a single future period, but about a stream of discounted consumption decisions influenced by both realised and expected investment gains. The natural way of setting up such problems is within a broader expected utility framework, such as Samuelson (1969). However for such situations there are well-known problems with EUT’s limitations in distinguishing between risk and time preferences.
Appendix B.
Empirical evidence for ergodicity economics

Meder et al. (2021)

Meder et al. (2021) is an attempt to test the predictions made by EE, EUT, and Prospect Theory. Participants had to choose between 312 pairs of 50-50 lotteries, and were told that a random selection of 10 of their choices would be realised and applied to an initial wealth of 1000DKK (about $155) they had been endowed with. They would then receive this final amount. Each participant did this twice: on one day they faced additive dynamics, so each lottery was of the form ‘gain \(x\) or lose \(y\); on the other they faced multiplicative dynamics, so each lottery was of the form ‘increase your endowed wealth by \(x\%\) or decrease it by \(y\\%\).’

To learn about the dynamics, they had a training session on each day where they saw the effect the gambles had on their wealth lottery-by-lottery; however they did not get this form of updating in the actual experiment. Their choices were then analysed and used to estimate an isoelastic utility function for them on each day:

\[
u(w) = (w^{1-\rho} - 1)/(1 - \rho).
\]

They claim that EE predicts \(\rho = 0\) for all participants on the additive day and \(\rho = 1\) for all participants on the multiplicative day; EUT places no restrictions on \(\rho\) other than that each participant’s value of \(\rho\) should be the same on both days. The experiment found that EE’s predictions were met, and that EUT’s were not.

The study received criticism from several sources, most notably in the supplement to Doctor et al. (2020). They noted that EUT was used in a static rather than dynamic way “as if intermediate outcomes were actually received”; there was no analysis of the actual probability distribution faced over the whole experiment; and outcomes were ambiguous but no adjustment was made for ambiguity aversion, or for aversion stemming from the complexity of the probability calculus (Doctor et al. 2020, Supplement pp. 7–9). This critique is supported by our analysis above: EUT is capable of evaluating decisions over time, and ignoring the dynamic nature of the problem may lead to an incorrect prediction.

The primary response to this criticism has been that cognitive constraints prevent individuals in this experiment from ‘looking ahead’ even to the next gamble, and thus the myopic EUT used is appropriate. Meder et al. (2021) states “Whilst it is possible in principle to update one’s expected wealth as a function of the decisions already made, this is computationally implausible, especially under the
demanding cognitive constraints of the task. To compute expected wealth for a given trial, past choices must be recalled and integrated over all possible outcomes. This integration quickly becomes computationally implausible, especially for the multiplicative condition which must consider all the possible wealth trajectories up to the given point in time” (Meder et al. 2021, 11). In their reply to Doctor et al. (2020), published as a supplement to Meder et al. (2021) they state “If subjects are limited by experimental design, unable to compute outcomes of the future, the dynamic models, when faced with this task, effectively make the same predictions as the static models” (Meder et al. 2021, S4 p. 1).

The claim about computational limitations is therefore central, as the authors seem to have conceded that they are not using EUT entirely appropriately. There is no reason to disbelieve the neuroscience, but it does not follow that the defence holds. To see this, note that Meder et al. (2021) implicitly assume a binary: either participants have full knowledge of the structure of the experiment and unlimited processing power, or they are entirely myopic. There is surely a middle ground. For example, participants are certainly unable to calculate the moments of the probability distribution and update these as they move through the experiment, but they may well have intuitive estimates of them. They might consider the learning stage they experienced and remember it seemed quite generous, and therefore expect the mean gain per realised lottery would be somewhat above zero (Meder et al. 2021, 6). They might consider that their wealth seemed to jump around a lot—or a little—in the training session, and consider the implications of this when faced with a risky lottery. Similarly, skewness and kurtosis may have been considered with this informal but not uninformative method. Empirical work suggests that individuals are capable of making these inferences, although they are certainly less accurate than formal calculations. Moreover, Meder et al. (2021) implicitly assume participants can tell the difference between the days, or they would have no reason to make different choices on them, which should raise the question of whether applying myopic EUT is sensible.

If participants can know something about the distribution of outcomes, then the analysis changes. Each choice they faced is best understood as a choice between two lotteries, each of which might be applied to an uncertain wealth level determined by the interaction of their experimentally-endowed wealth with 9 other lotteries. Both the additive and multiplicative days were generous, so participants

9 For example, Goldstein and Rothschild (2014) test how individuals can describe probability distributions, including the mean, and Nisbett et al. (1983) suggest that people can reason quite well statistically when primed to do so. There is also a large quantity of work showing how far from perfection individuals can be, such as Kahneman, Slovic, and Tversky (1982). Even if they were very inaccurate, we argue that the participants in Meder et al. (2021) would likely be have been able to make the distinctions discussed above.
had reason to believe that this wealth level would probably substantially exceed their 1,000DKK starting wealth. This alone implies that myopic EUT is misleading. On the additive day, experimental wealth is underestimated, and so a risk-averse participant with isoelastic utility would perceive the same lottery as less risky than the experimental analysis assumes. On the multiplicative day, experimental wealth is underestimated which also tends to overestimate the riskiness of the lottery faced—but there is a countervailing force, because the size of the experimental wealth the multiplicative lottery is applied to is much larger, and thus the riskiness of the lottery is underestimated. Note that if the participant did only possess their experimentally-endowed wealth and had isoelastic utility this would not matter because the relative amount of risk is independent of the wealth on which the lottery is applied. However, because participants had external wealth (which the experimenters did not gather information on) this result does not hold. Furthermore, ignoring external wealth (which is likely to be far in excess of experimental wealth, plausibly by at least two orders of magnitude) also biases the estimates, as lotteries appear more risky than they really are.

Due to a misunderstanding of EUT, Meder et al. (2021)’s estimation process is flawed. This is just as well for EE’s supporters as it is very unclear that, had the estimates been unbiased, the results would have supported EE. This is because, as discussed above, decisions on the additive day are independent of each other. The growth dynamic outside of the experiment should therefore be taken into account—and this is likely to be multiplicative (since the individuals could bank their winnings, or invest them in the stock market, etc.).

Vanhoyweghen et al. (2022)

Vanhoyweghen et al. (2022) is a response to some of the criticisms of Meder et al. (2021), but suffers from distinct problems. Eighty-one participants faced a series of eighty choices between two 50-50 lotteries, of which forty were in an additive setting and forty were in a multiplicative setting. These lotteries were applied to a starting ‘wealth’ of 1,000. Unlike Meder et al. (2021), the participants faced all the lotteries in a single session, and were shown the lotteries explicitly, rather than associating them with fractals. Crucially, the choices were designed such that there was a ‘safer’ and ‘riskier’ lottery: both lotteries had the same expected

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10. Peters and Gell-Mann (2016, 5) agree: “Wealth is often better modelled with multiplicative dynamics.”
11. Unlike Meder et al. (2021) the wealth is purely within the experiment—it cannot be used or withdrawn by the participants.
value but the riskier one had a greater variance. (There were also ten “no-brainer” choices where one lottery is clearly better than the other, i.e., first-order stochastic dominance.) This experimental design was intended to discriminate between EUT and EE behaviour: “this set-up allows for easy differentiation between optimal and non-optimal behaviour according to time-average growth. The bets [lotteries] had the same expected value and time-average growth in the additive setting, which means none of the bet should dominate. However, bets with lower variance and equal expected value -the safer bets- always yielded a higher time-average growth than their more risky counterparts in the multiplicative setting” (Vanhoywhegen et al. 2022, 3).

They assert that their results “lend support to the theory that intuitive human decision-makers behave differently depending on whether their environment is ergodic or non-ergodic. In agreement with the findings of ergodicity economics, intuitive decision makers tend to optimize the time average of their wealth over the expected values” (Vanhoywhegen et al. 2022, 5).

There are three distinct problems with this experimental design. Firstly, we have seen above that approaching a dynamic problem as if each stage is a static decision is an incorrect application of EUT. Similarly to Meder et al. (2021), they argue that cognitive constraints remove the need for a dynamic analysis:

[W]e opted to use the risk argument (MC1) rather than extensions on utility theory which takes multiple periods into account. We believe this argument to be appropriate because respondents could not envisage the capital they would have at the end of the experiment and as such could not use this terminal capital as a heuristic within our experiment; they simply had too little information to increase their decision algorithm to multiple periods. Respondents were kept in the dark of the bets that were to come, including the amount of capital they possessed after each decision, bet outcomes, and the order used to determine terminal wealth, which removes the need for dynamic programming. (Vanhoywhegen et al. 2022, 5)

Curiously, they refer to Samuelson (1975; originally 1969) to support this claim, although that paper makes no such argument. In any case, it fails for precisely the same reasons we present in our analysis of Meder et al. (2021) above.

Secondly, even if we were to take the claim that the lotteries can be evaluated statically at face value, the experiment cannot discriminate between EUT and EE. When a pair of lotteries have the same expected value but differing variances, we can say that the ‘safer’ lottery with lower variance second-order stochastically dominates the ‘riskier’ lottery with higher variance. Under EUT, any risk-averse agent will avoid a second-order stochastically dominated lottery, whilst a risk-neutral agent will be indifferent between them. Therefore if participants are well-
described as risk-averse expected utility maximisers, we would expect no difference between their results and the predictions of EE.

Finally, the experiment’s reward structure also muddies its conclusions. Participants did not benefit from their experimentally-endowed wealth in itself, but instead received a fixed prize if they were one of the six participants with the highest endowed wealth at the end of the experiment. If all participants followed the ‘safer’ strategy, then by symmetry their probability of receiving a prize would be $6/81$. If they switched one lottery to its ‘riskier’ counterpart, they would improve or worsen their relative position, each with probability $0.5$. However the payoff is asymmetric: improving their position plausibly increased their probability of winning a prize more than worsening it decreased their probability of winning a prize, due to the improbability of them winning a prize under the original strategy. Therefore participants were incentivised to maximise their chance of winning the prize, which is distinct from maximising, in expectation, their experimental wealth. We note that they try to deal with this problem by providing six prizes—“In order to motivate respondents and mitigate the excess risk-taking linked to winner-take-all games, six prizes could be won” (Vanhoywhegen et al. 2022, 8)—but as we show this is insufficient.

A misunderstanding of EUT has resulted in an experiment that cannot discriminate between EUT and EE, and therefore we cannot accept the claim that this experiment offers empirical support for EE.

References


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