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Toward Bubble Clarity: A Comment on Miao and Wang[1]

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LINK TO ABSTRACT

Imagine that a few economists get together to discuss some topic, say "equilibrium." Arrow or Debreu may have general equilibrium in mind. Nash may have Nash equilibrium in mind. Others may have different concepts in mind, such as Bayesian Nash equilibrium, correlated equilibrium, Markov perfect equilibrium, recursive competitive equilibrium, sequential equilibrium, subgame perfect equilibrium, and so on. To have a productive scientific debate, it is important to communicate what we mean by words central to the discussion.

Now, change "equilibrium" to "bubble." There is significant polysemy of this word in the economic literature. The most common and widely accepted definition, which we will adhere to and elaborate upon below, is a situation in which "asset prices do not reflect fundamentals" (Stiglitz 1990), or in other words, the asset price (P) exceeds its fundamental value (V) defined by the present value of dividends (D). The condition $P > V$ could arise for various reasons. For instance, if agents hold heterogeneous beliefs or asymmetric information, the fundamental value V need not be common across agents (whereas the market price P is common across agents), and hence it could be $P > V$ for some or even all agents (Scheinkman and Xiong 2003; Fostel and Geanakoplos 2012; Barlevy 2014; Allen et al. 2022).

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In infinite-horizon general equilibrium models, $P > V$ could hold even with rational expectations (common beliefs and complete information) because in equilibrium, agents believe that they can resell in the future the overpriced asset to other agents, i.e., there is a speculative motive. This class of models is generally referred to as *rational bubbles*; such models were introduced and studied by seminal papers (Samuelson 1958; Bewley 1980; Tirole 1985; Scheinkman and Weiss 1986; Kocherlakota 1992; Santos and Woodford 1997), as well as the large subsequent literature (reviewed in Martin and Ventura 2018 and Hirano and Toda 2024a). In what follows, to prevent ambiguity, we refer to rational bubbles as just "RB."[4] Put simply, as we explain below under "Formal definitions," RB means speculation: investors rationally purchase the asset at a high price for the purpose of reselling in the future, rather than (or in addition to) receiving dividends.

Some authors (and the popular press) use the word *bubble* loosely to describe booms and busts in asset prices. For instance, several empirical papers including Òscar Jordà et al. (2015) define a bubbly episode as a deviation from the trend that is larger than some pre-specified threshold. In the time series econometrics literature (Phillips et al. 2015; Phillips and Shi 2020), a bubble is defined as an explosive dynamic of the price-dividend ratio. In monetary theory, bubble is synonymous to liquidity premium (Lagos et al. 2017). As such, it is natural that there are diverse views and definitions regarding asset price bubbles, and it is important to understand asset prices from a broader perspective.

In the RB literature, as we explain below under "Fundamental difficulty in attaching rational bubbles to dividend-paying assets," there is a well-known fundamental difficulty in proving the existence of speculation attached to an asset with positive dividends. Indeed, in workhorse macro-finance models, there is no aspect of speculation in asset prices in equilibrium. Therefore, the macro-finance literature including RB has regarded attaching bubbles to dividend-paying assets in a natural way as a longstanding and important open question.[5]

The article by Jianjun Miao and Pengfei Wang (2018) published in *American Economic Review* is ambitious, as it claims to "provide a theory of rational stock price bubbles" (abs.). As it is, their paper gives the impression that they have proved the existence of rational bubbles (RB) attached to real assets and resolved the

4. The scientific term *rational bubble* is well established, as evident by the titles of Santos and Woodford (1997), "Rational Asset Pricing Bubbles," and Martin and Ventura (2018), "The Macroeconomics of Rational Bubbles: A User's Guide."

5. On this point, we thank José Scheinkman and Nobuhiro Kiyotaki for pointing out the limitations of models with zero dividends and teaching us how difficult and how valuable it is to prove the existence of RB attached to real assets with positive dividends. Examples of RB attached to dividend-paying assets include Wilson (1981), Olivier (2000), Bosi et al. (2018), and Hirano and Toda (2025). See Hirano and Toda (2024b) for a review of the recent development of RB attached to real assets.

fundamental difficulty discussed above. Indeed, at various places, a number of papers on RB are cited and compared, including seminal papers by Paul Samuelson (1958) and Jean Tirole (1985). For instance, by citing basic theory papers on RB, Miao and Wang state:

> Some studies (e.g., Scheinkman and Weiss, 1986; Kocherlakota, 1992, 2008; Santos and Woodford, 1997; Hellwig and Lorenzoni, 2009) have found that infinite-horizon models of endowment economies with borrowing constraints can generate rational bubbles. Unlike this literature, our paper analyzes a production economy with stock price bubbles attached to productive firms. (Miao and Wang 2018, 2595)

This passage suggests that Miao and Wang view their marginal contribution as establishing RB in an infinite-horizon production economy, as opposed to endowment economies studied in the earlier literature. Concerning the bubble existence condition, Miao and Wang state

> We can also show that the bubbleless equilibrium is dynamically efficient in our model. […] Thus, the condition that the economy must be dynamically inefficient in Tirole (1985) cannot ensure the existence of bubbles in our model. (Miao and Wang 2018, 2607)

This passage suggests that Miao and Wang think that the existence conditions of RB are different between their model and Tirole's (1985). Furthermore, in a subsequent article coauthored with Feng Dong, Miao and Wang write:

> Introducing dividends or rents will complicate our analysis without changing our key insights. See Miao and Wang (2018) and Miao, Wang, and Xu (2015) for models of rational bubbles attached to assets with dividends or rents. (Dong, Miao, and Wang 2020, S69 n.4)

There the authors explicitly state that Miao and Wang (2018) is RB.

In the literature, the model of Miao and Wang (2018) appears to have been understood as RB. According to our systematic literature search detailed in Appendix A, there are 74 papers that are mainly about the theory of bubbles (broadly defined) and cite Miao and Wang (2018) in the context of bubbles. Among these 74 papers, 68 cite it in the context of RB specifically. However, Miao and Wang (2018) do not prove the existence of RB in their model. In their Equation (18), they show that the stock price of firm j given capital K^j equals $V_t(K^j) = Q_t K^j + B_t$, where they "interpret $Q_t K^j$ as the fundamental value of the firm" and "interpret B_t as a bubble component" (Miao and Wang 2018, 2601). This begs the question of whether their model is indeed RB, as claimed by Miao and Wang (2018) and Dong, Miao, and

Wang (2020) and cited by the large subsequent literature.

In the present paper, we prove in Proposition 1 that RB do not exist in the model of Miao and Wang (2018). For this purpose, in Lemma 2 we extend the Bubble Characterization Lemma of Luigi Montrucchio (2004) to a continuous-time setting, which could be viewed as our (modest) technical contribution.[6] Stock prices in Miao and Wang (2018) reflect fundamentals and therefore do not contain the aspect of speculation. Therefore, it is incorrect to understand the model of Miao and Wang (2018) as dealing with RB as speculation pioneered by Samuelson (1958). As we discuss below under "Related literature," it is more appropriate to interpret Miao and Wang (2018) as a model of multiple fundamental equilibria, as in ordinary macro-finance models, in which the asset prices are always equal to the present discounted value of dividends.

To make these points clear, this paper proceeds as follows. After this introduction, following the textbook treatment of Jianjun Miao (2014, §13.6), we review the precise and standard definition of "rational bubbles" (RB) in a self-contained way so that the general audience can follow. RB is a situation in which the no-bubble condition is violated, and hence the asset price exceeds its fundamental value. In brief, it is a bubble as speculation backed by nothing. Next we discuss the simple yet powerful Bubble Characterization Lemma of Montrucchio (2004), which provides a necessary and sufficient condition for the existence of RB in economies without aggregate uncertainty. We then explain the fundamental difficulty in attaching RB to dividend-paying assets following the results of Manuel Santos and Michael Woodford (1997). Following that is the main section of this paper, where we prove in Proposition 1 the nonexistence of RB in the Miao and Wang (2018) model. Thereafter we discuss the related literature and conclude.

Open letter to Miao and Wang

To support a constructive scientific debate, we conclude the introduction with three queries addressed to Miao and Wang to address the most important issues we raised through the review process of an earlier draft (Hirano and Toda 2024c) at *American Economic Review*.[7] The three queries presuppose our definition of "rational bubble" given in the next section.

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6. As the proofs of Lemma 2 and Proposition 1 are straightforward, we received criticisms such as our paper lacks novelty. It is easy to criticize a trivial idea after seeing it demonstrated. In our opinion, the fact that the nonexistence of RB in the Miao and Wang (2018) model remained unnoticed for more than a decade (their working paper circulated since 2011) is evidence that the idea was ex ante nontrivial. Furthermore, for the development of the literature, it is important to show that a paper that has long been considered to be RB was, in fact, not.

7. Their report (**link**) and our response (**link**) are publicly available.

1. Are Lemmas 1 and 2 correct? Yes or No. (If No, provide a mathematical, not verbal, argument for why.)
2. Is Lemma 2 applicable in the context of the Miao and Wang (2018) model? Yes or No. (If No, provide a mathematical, not verbal, argument for why.)
3. Is Proposition 1 correct? Yes or No. (If No, provide a mathematical, not verbal, argument for why.)

Rational bubbles as speculation

Formal definitions

The formal definition of rational bubbles (RB) was given by Santos and Woodford (1997). Here we follow the textbook treatment of Miao (2014, §13.6) nearly verbatim. Consider an infinite-horizon economy with a homogeneous good and time indexed by $t = 0, 1, \ldots$.[8] Let π_t denote a state price deflator.[9] For instance, in a deterministic economy, π_t is the date-0 price of a zero-coupon bond with maturity t. Consider an asset with infinite maturity that pays dividend $D_t \geq 0$ and trades at ex-dividend price P_t, both in units of the time-t good. Then the no-arbitrage asset pricing equation[10] is given by

$$\pi_t P_t = E_t\big[\pi_{t+1}\big[P_{t+1} + D_{t+1}\big]\big]. \qquad (1)$$

Solving this equation forward by repeated substitution (and applying the law of iterated expectations) yields

8. The continuous-time case is briefly discussed in Appendix B.

9. If markets are incomplete, which is considered in Santos and Woodford (1997), π_t need not be unique. The subsequent discussion holds regardless of market completeness.

10. In specific models, this no-arbitrage condition must be derived from individual optimization problems, which usually comes from the first-order condition. For instance, in Hirano and Stiglitz (2024), land can be used as a means of savings and as collateral for borrowing, in which case a land collateral premium arises. They derive the no-arbitrage equation in a consistent manner with individual optimization and show that land prices can be written in the same form as our equation (2) and the collateral premium is included in the discount rate (see Hirano and Stiglitz 2024, Appendix O.4). They show that higher land prices due to the increased land collateral premium is different from rational land price bubbles as speculation backed by nothing. It is important to note that the discussion following our equation (1) is model-free.

$$\pi_t P_t = E_t \sum_{s=t+1}^{T} \pi_s D_s + E_t[\pi_T P_T]. \tag{2}$$

Because all terms are nonnegative, the sum in (2) from $s = t + 1$ to $s = T$ is (i) increasing in T and (ii) bounded above by $\pi_t P_t$, so it converges almost surely as $T \to \infty$. Therefore, the fundamental value of the asset

$$V_t := \frac{1}{\pi_t} E_t \sum_{s=t+1}^{\infty} \pi_s D_s \tag{3}$$

is well-defined, and letting $T \to \infty$ in (2), we obtain $P_t = V_t + B_t$, where we define the *asset price bubble* as

$$B_t := \lim_{T \to \infty} \frac{1}{\pi_t} E_t[\pi_T P_T] \geq 0. \tag{4}$$

That is, an asset price bubble is equal to the difference between the market price of the asset and its fundamental value (i.e., the present value of dividends). By definition, there is no bubble at time t if and only if

$$\lim_{T \to \infty} E_t[\pi_T P_T] = 0. \tag{5}$$

This is the mathematical formalization of the idea explained in Stiglitz (1990). We refer to (5) as the no-bubble condition.[11]

Three remarks are in order. First, the economic meaning of the bubble component B_t in (4) is that it captures a speculative aspect, that is, agents buy the asset now for the purpose of resale in the future, rather than for the purpose of receiving dividends. When the no-bubble condition (5) holds, the aspect of speculation becomes negligible and asset prices are determined only by factors that are backed in equilibrium, namely future dividends. On the other hand, if

11. Samuelson (1958) does not explicitly state the no-bubble condition (5) but correctly understands that no-arbitrage alone does not pin down asset prices, writing: "We never seem to get enough equations: lengthening our time period turns out always to add as many new unknowns as it supplies equations" (470). The early theoretical literature such as Magill and Quinzii (1994; 1996), Santos and Woodford (1997), and Montrucchio (2004) refers to (5) as the "transversality condition". In our previous papers (Hirano and Toda 2024a; 2025), we followed this terminology but faced push-backs from the general audience who are not familiar with the distinction between this transversality condition *for asset pricing* and the transversality condition *for optimality* in infinite-horizon optimal control problems (Toda 2025, §15.3), which are completely different concepts. To avoid unnecessary controversy, we use "no-bubble condition."

$\lim_{T\to\infty} E_t\left[\pi_T P_T\right] > 0$, equilibrium asset prices contain a speculative aspect. Second, if $D_t = 0$ for all t (Samuelson (1958)'s case of *money* or *pure bubble*), the fundamental value of the asset is zero and there is only the aspect of speculation. In this case, $P_t > 0$ if and only if $\lim_{T\to\infty} E_t\left[\pi_T P_T\right] > 0$, i.e., the current price of the asset will be positive in equilibrium if and only if one expects that one will be able to resell the asset at a positive price in the future. The definition of RB is the same for money and for real assets yielding positive dividends: it is the speculative component of the asset price. Third, although the definition is the same, there is a discontinuity in proving the existence of RB between the cases with $D_t = 0$ and $D_t > 0$. In other words, as we explain below, it is well known from Santos and Woodford (1997) that there is a fundamental difficulty in generating a bubble attached to an asset with $D_t > 0$.

Bubble Characterization Lemma

To prove the existence of RB, we need to prove $P > V$, or equivalently, verify the violation of the no-bubble condition (5). For an asset that pays no dividends ($D = 0$, pure bubble), because the fundamental value is necessarily zero, showing $P > 0$ suffices. However, for dividend-paying assets, i.e., real assets such as stocks, land, and housing, the verification of the violation of the no-bubble condition is not easy because it is cumbersome to calculate the state price deflator π_t. Fortunately, in economies without aggregate uncertainty, there is a very simple characterization due to Montrucchio (2004).

Lemma 1. Bubble Characterization (Montrucchio 2004, Proposition 7). *In an economy without aggregate uncertainty, if $P_t > 0$ for all t, the asset price exhibits a rational bubble (RB) if and only if $\sum_{t=0}^{\infty} D_t / P_t < \infty$.*

Hirano and Toda (2025) give a simple proof of the Bubble Characterization Lemma and describe its many applications.

Fundamental difficulty in attaching rational bubbles to dividend-paying assets

A primordial though overlooked implication of the Bubble Characterization Lemma is that if the price-dividend ratio P_t / D_t (or the dividend yield D_t / P_t) converges to a positive constant, then necessarily $\sum_{t=0}^{\infty} D_t / P_t = \infty$, so there is no RB by Lemma 1. This implies that in a model with dividend-paying assets, irrespective of the model setup, RB as speculation can never occur if the price-dividend ratio (or the dividend yield) converges to a positive constant. In particular, in models where $(P_t, D_t) = (P, D)$ is constant, RB can arise only if $D = 0$, i.e., money or pure bubble. We can easily see this result even without appealing to Lemma 1. Consider an asset with price $P > 0$ and dividend $D > 0$. Then the gross risk-free rate is $R = (P + D) / P > 1$, so the fundamental value of the asset is

$$V = \sum_{s=1}^{\infty} R^{-s} D = \frac{D / R}{1 - 1 / R} = \frac{D}{R - 1} = \frac{D}{\frac{P+D}{P} - 1} = P.$$

Since $P = V$, there is no RB.

More generally, RB cannot arise if dividends consist of a non-negligible fraction of aggregate endowments. This fundamental difficulty of attaching RB to dividend-paying assets follows from Santos and Woodford (1997), who show in their Theorem 3.3 that if the present value of the aggregate endowment is finite, then the price of an asset in positive net supply or with finite maturity equals its fundamental value. Furthermore, their Corollary 3.4 (together with their Lemma 2.4) shows that, when the asset pays nonnegligible dividends relative to the aggregate endowment, RB as speculation are impossible.[12]

The bubble impossibility results of Santos and Woodford (1997) have been understood as a fundamental difficulty in attaching bubbles to dividend-paying assets. Perhaps due to this fundamental difficulty, the RB literature (for reviews, see Martin and Ventura 2018; Hirano and Toda 2024a) has almost exclusively studied pure bubble models with $D_t = 0$ and considered attaching bubbles to dividend-paying assets in a natural way as a longstanding and important open question. Therefore, if one claims to prove the existence of RB attached to an asset with $D_t > 0$, it is the author's responsibility to verify so in a manner consistent with Santos and Woodford (1997) and Montrucchio (2004). Otherwise, it does not mean one has proved it.

Nonexistence of rational bubbles in Miao and Wang (2018)

Miao and Wang (2018) (henceforth MW) claim to "provide a theory of rational stock price bubbles" in an equilibrium model in relation to financial conditions (MW, abs.). Here is a brief model description. There is a continuum of firms maximizing shareholder value, and investment opportunities arrive stochastically according to independent Poisson processes. Upon the arrival of an investment opportunity, a firm may invest subject to a collateral constraint that

12. See Hirano and Toda (2024a, §3.4) for an accessible discussion of these results.

involves the firm value. Otherwise, the model is a standard neoclassical growth model (representative households, neoclassical production function, no capital adjustment costs, etc.).[13]

MW (2601 eq. 18) show that the value of a firm with capital K_t (stock price) equals $V_t(K_t) = Q_tK_t + B_t$, where $Q_t > 0$ and $B_t \geq 0$ are coefficients depending only on time. Note that because investment opportunities arrive to firms stochastically, a firm with no capital at present could have a positive value $V_t(0) = B_t$ in anticipation of the arrival of future investment opportunities. Importantly, MW (2601) "interpret" Q_tK_t as the fundamental value and B_t as the bubble. Based on the way Miao and Wang write their paper, it is clear that they view their model as RB, and in fact Dong, Miao, and Wang (2020, n.4) explicitly states so. However, using the Bubble Characterization Lemma, it is straightforward to show that in the MW model, the stock price equals the fundamental value and hence there is no speculative aspect.

Proposition 1. *Rational bubbles (RB) do not exist in the model of Miao and Wang (2018).*

Proof. In the MW model, there is a continuum of firms of unit measure subject to only idiosyncratic risks. Let $P_{it} := Q_tK_{it} + B_t$ be the stock price of firm i at time t, and let V_{it} be the expected present discounted value of future dividends (fundamental value). Similarly, let $P_t := Q_tK_t + B_t$ be the price of the aggregate stock market (where K_t is aggregate capital) and V_t be the present discounted value of future aggregate dividends (fundamental value). By the discussion under "Formal definitions" above, we have $P_{it} - V_{it} \geq 0$. Aggregating across firms, we have $P_t - V_t \geq 0$, with equality if and only if $P_{it} = V_{it}$ for i almost surely. Therefore, to prove the nonexistence of rational stock price bubbles in individual firms, it suffices to prove the nonexistence of RB in the aggregate stock market.

Because there is no aggregate uncertainty, aggregate dividend is paid out continuously according to $dF_t = D_tdt$ in the notation of Lemma 2 in Appendix B. Furthermore, the aggregate stock price P_t and the dividend D_t are deterministic and converge to constants $P, D > 0$.[14] In particular, we have $\int_0^\infty D_t / P_t dt = \infty$. As the MW model features a representative household and the aggregate stock is in positive net supply, the equilibrium portfolio choice is interior. Therefore, the no-arbitrage condition holds using this household's stochastic discount factor[15] and hence Lemma 2 is applicable. Therefore, there exist no RB in either the aggregate stock market or individual firms. *QED.*

13. Because Miao and Wang (2018) provide a micro-foundation for the collateral constraint and derive the continuous-time model by carefully taking the limit of a discrete-time model, the exposition is rather involved. See Sorger (2020) for a concise model description.

14. The positivity of dividend is discussed in the displayed equation on page 2608 of MW.

15. MW assume risk-neutral households, but this assumption is inessential.

The problem with Miao and Wang (2018) is that their logic flows in the reverse direction to RB models, yet they write their paper as if they have resolved the "fundamental difficulty in attaching rational bubbles to dividend-paying assets" described above. To clarify, in standard RB models, as described under "Formal definitions" above, the flow of logic is that (i) the asset pricing equation (1) is derived from individual optimization, and then (ii) whether there is RB or not (whether $P > V$ or $P = V$) is verified based on the no-bubble condition (5). In contrast, in MW, the flow of logic is that (i') the bubble component B_t is defined based on an "interpretation" of their own, and then (ii') the differential equation that B_t satisfies (which is Equation (20) in MW) is interpreted as an asset pricing equation. What concerns us is that MW repeatedly use the word "rational" (20 times in total) and compare their results to the RB literature as quoted in the introduction, without acknowledging the fact that the scientific definition of RB is completely different. The crucial point is that if one claims to prove the existence of RB attached to an asset with $D_t > 0$ following the formal definition given in this paper, the question is simple and clear: one needs to verify the violation of the no-bubble condition given by (5), or equivalently, the consistency with the Bubble Characterization Lemma 1. It is the author's responsibility in the first place to show whether the no-bubble condition is satisfied or not. MW never verified this in their model. In other words, MW claim to have solved an important problem in the literature by changing the problem itself.

Related literature

Several papers study models like MW. In these papers, for justifying bubbles, an expression like "we interpret B as a bubble" appears, without reference to the formal definition or the economic meaning of RB. Table 1 lists these expressions in MW (whose working paper circulated since 2011) and the subsequent follow-up papers that we found based on our systematic literature search detailed in Appendix A.

These papers appear to have been understood as models of RB attached to real assets (Appendix A). However, by Proposition 1, asset prices in these models reflect fundamentals, regardless of the interpretation of the term B. In other words, in all papers in Table 1, no RB exist. Thus, the way Miao and Wang (2018) write their paper is misleading as they claim the existence of a bubble as if it were RB.

It is more appropriate to interpret Miao and Wang (2018) and others as multiple fundamental equilibria in asset pricing models, where there are two steady states, one with high stock prices and the other with low stock prices. In both steady states, stock prices always reflect fundamentals, but self-fulfilling expectations determine which steady state is reached.[16] In fact, Miao and Wang (2015, 772) state "one may also interpret it as a self-fulfilling component or a belief component if one wants to avoid using the term 'bubble.'" In empirical research, it is challenging to distinguish between bubbles and time-varying fundamental values (see Gürkaynak 2008 for a survey). The MW model captures changes in fundamental values and hence it is not useful for the identification of asset bubbles in empirical studies.

TABLE 1. Typical expressions in models of $V(K)=QK+B$

Paper	Expression	Page
Miao and Wang (2012)	"we require $b_{it} > 0$ and interpret it as a bubble"	85
Miao and Wang (2014)	"We interpret $B_{it} > 0$ as the bubble component"	157
Miao and Wang (2015)	"We may interpret B_t as a bubble component"	772
Miao, Wang, and Z. Xu (2015)	"we may interpret this component as a bubble"	608
Miao, Wang, and L. Xu (2016)	"$B_t > 0$ for all t and interpret it as a bubble"	283
Miao and Wang (2018)	"we interpret B_t as a bubble component"	2601
Chevallier and El Joueidi (2019)	"$b_t \neq 0$ is the bubble term"	122
Sorger (2020)	"$q(t) \neq 0$ is said to contain a bubble"	525
He (2021)	"the second term is viewed as the bubble"	251
Ikeda (2022)	"interpreted as a liquidity bubble"	1579

Our proof of Proposition 1, the nonexistence of RB in the MW model, is arguably indirect, as we appeal to the Bubble Characterization Lemma to prove the nonexistence of RB in the aggregate stock market without referring to individual stock prices or dividends. Proving directly that $P_{it} = V_{it}$ for individual stocks is not easy (and explains why this result remained unnoticed for more than a decade), as investment opportunities arrive to firms randomly and hence both the stock price P_{it} and dividend D_{it} are stochastic. However, what matters is whether the proof is mathematically correct or not; the proof methodology is irrelevant. That said, in a recent working paper, Takeo Hori, Ryonghun Im, and Hiroshi Nakaota (2024, Proposition 4) directly verify that $P = V$ in a simplified model of MW without idiosyncratic shocks, i.e., stock prices of the form $QK + B$ are entirely determined by the present discounted value of dividends and the no-bubble condition holds.

16. Azariadis (1981), Cass and Shell (1983), and Farmer (1999) have produced a large literature that studies the effects of self-fulfilling expectations on macroeconomic outcomes including asset prices, which is clearly important. Miao and Wang (2018) and others would belong to this large literature.

Concluding remarks

There are diverse views and definitions regarding bubbles and fundamental values. We have an open mind, and we welcome various approaches to bubbles such as based on rationality, heterogeneous beliefs, asymmetric information, self-fulfilling expectations, econometrics, empirical considerations, or others, as long as definitions are clear and consistent throughout the paper. In our opinion, a significant merit of the classical "rational bubble" (RB) approach pioneered by Samuelson (1958), Bewley (1980), Tirole (1985), Scheinkman and Weiss (1986), Kocherlakota (1992), Santos and Woodford (1997), and outlined in this article is that the definition is precise, model-free, and has an important economic meaning, i.e., speculation backed by nothing.

C. S. Lewis (1960, 2–3) said that "language…contains, with all its defects, a good deal of stored insight and experience. If you begin by flouting it, it has a way of avenging itself later on. We had better not follow Humpty Dumpty in making words mean whatever we please."

"Rational bubble" is a scientific term that has a specific definition, and we had better not flout it. Our theoretical contribution is that we mathematically proved that the "bubble" in Miao and Wang (2018) is different from the classical "rational bubble." We hope that our paper clarifies the confusion in the literature and that the science of bubbles prospers, without ever imploding.

Appendix A. Systematic literature search

We used Google Scholar to identify papers that cite Miao and Wang (2018) (MW) or its working paper version titled "Bubbles and Credit Constraints." As of the time of search (July 2024), MW was cited 198 times and the working paper version 125 times. Of these citations, we only retained published journal articles written in English and removed duplicate citations, resulting in 125 citations. We then checked each paper to see how MW is cited. Of these 125 papers, 74 are mainly about the theory of bubbles (broadly defined, not necessarily rational bubbles), 96 cite MW in the context of bubbles (as opposed to other contexts such as credit constraints, investment, multiple equilibria, etc.), and 71 cite MW in the context of *rational bubbles* (RB) specifically. (We regard the citation as RB if the paper explicitly mentions "rational bubble" or cites MW along with other papers on RB.) All 74 papers that are mainly about the theory of bubbles (broadly defined) cite MW in the context of bubbles. Among these 74 papers, 68 cite MW in the context of RB specifically. For more details, see the spreadsheet (**link**).

Appendix B. Bubble characterization in continuous time

This appendix extends Lemma 1 to continuous time.

First, consider the discrete-time setting in our “Formal definitions” section but without aggregate uncertainty.

If the asset is risk-free, taking the unconditional expectations of (1) and setting $q_t = E[\pi_t] > 0$ (which equals the date-0 price of a zero-coupon bond with maturity t), we obtain

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}). \qquad (6)$$

Then by the same argument as in the “Formal definitions” section and using $q_0 = 1$, we obtain

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T, \qquad (7)$$

and there is no bubble if the no-bubble condition $\lim_{T\to\infty} q_T P_T = 0$ holds.[17]

Now consider the continuous-time model without aggregate uncertainty. Let $F_t \geq 0$ be the cumulative dividend payout of an asset up to time t (so $t \longmapsto F_t$ is weakly increasing and right-continuous), and P_t be its ex-dividend price. In a small time interval $(t, t+\Delta]$, the dividend is $F_{t+\Delta} - F_t$. Therefore, the continuous-time counterpart of the no-arbitrage condition (6) is

$$q_t P_t = q_{t+\Delta}(P_{t+\Delta} + F_{t+\Delta} - F_t).$$

Subtracting $q_{t+\Delta} P_{t+\Delta}$ from both sides and taking the limit $\Delta \to 0$, we obtain

$$-d(q_t P_t) = q_t dF_t. \qquad (8)$$

Integrating both sides from $t = 0$ to $t = T$ and using $q_0 = 1$, we obtain

$$P_0 = \int_0^T q_t dF_t + q_T P_T, \qquad (9)$$

17. See Hirano and Toda (2024a, §2; 2025, §2) for details.

which is the continuous-time counterpart of (7). Multiplying both sides of (5) by π_t, taking the unconditional expectation, and letting $T \to \infty$, we see that the no-bubble condition is $\lim_{T\to\infty} q_T P_T = 0$. Under this condition, by (9), the asset price equals its fundamental value $V_0 := \int_0^\infty q_t dF_t$.

The following lemma provides a continuous-time counterpart of Lemma 1.

Lemma 2. Bubble characterization in continuous time. *In an economy without aggregate uncertainty, if* $P_t > 0$ *for all* t*, the asset price exhibits a rational bubble if and only if* $\int_0^\infty dF_t / P_t < \infty$.

Proof. Dividing both sides of (8) by $q_t P_t > 0$ and integrating from $t = 0$ to $t = T$, we obtain

$$\int_0^T \frac{dF_t}{P_t} = -\int_0^T \frac{d(q_t P_t)}{q_t P_t} = -\int_0^T d\log(q_t P_t) = \log(q_0 P_0) - \log(q_T P_T).$$

Rearranging terms, we obtain

$$q_T P_T = q_0 P_0 \exp\left(-\int_0^T \frac{dF_t}{P_t}\right).$$

Letting $T \to \infty$, the no-bubble condition $\lim_{T\to\infty} q_T P_T = 0$ holds if and only if $\int_0^\infty dF_t / P_t = \infty$. *QED*.

References

Allen, Franklin, Gadi Barlevy, and Douglas Gale. 2022. Asset Price Booms and Macroeconomic Policy: A Risk-Shifting Approach. *American Economic Journal: Macroeconomics* 14(2): 243–280.

Azariadis, Costas. 1981. Self-Fulfilling Prophecies. *Journal of Economic Theory* 25(3): 380–396.

Barlevy, Gadi. 2014. A Leverage-Based Model of Speculative Bubbles. *Journal of Economic Theory* 153: 459–505.

Bewley, Truman. 1980. The Optimum Quantity of Money. In *Models of Monetary Economies*, ed. John H. Kareken and Neil Wallace, 169–210. Federal Reserve Bank of Minneapolis (Minneapolis, Minn.).

Bosi, Stefano, Thai Ha-Huy, Cuong Le Van, Cao-Tung Pham, and Ngoc- Sang Pham. 2018. Financial Bubbles and Capital Accumulation in Altruistic Economies. *Journal of Mathematical Economics* 75: 125–139.

Cass, David, and Karl Shell. 1983. Do Sunspots Matter? *Journal of Political Economy* 91(2): 193–227.

Chevallier, Claire Océane, and Sarah El Joueidi. 2019. Capital Regulation and Banking

Bubbles. *Journal of Mathematical Economics* 84: 117–129.
Dong, Feng, Jianjun Miao, and Pengfei Wang. 2020. Asset Bubbles and Monetary Policy. *Review of Economic Dynamics* 37: S68–S98.
Farmer, Roger E. A. 1999. *Macroeconomics of Self-Fulfilling Prophecies*, 2nd ed. MIT Press.
Fostel, Ana, and John Geanakoplos. 2012. Tranching, CDS, and Asset Prices: How Financial Innovation Can Cause Bubbles and Crashes. *American Economic Journal: Macroeconomics* 4(1): 190–225.
Gürkaynak, Refet S. 2008. Econometric Tests of Asset Price Bubbles: Taking Stock. *Journal of Economic Surveys* 22(1): 166–186.
He, Sicheng. 2021. Growth, Innovation, Credit Constraints, and Stock Price Bubbles. *Journal of Economics* 133(3): 239–269.
Hellwig, Christian, and Guido Lorenzoni. 2009. Bubbles and Self-Enforcing Debt. *Econometrica* 77(4): 1137–1164.
Hirano, Tomohiro, and Joseph E. Stiglitz. 2024. Credit, Land Speculation, and Long-Run Economic Growth. *NBER Working Paper* 32479. National Bureau of Economic Research (Cambridge, Mass.).
Hirano, Tomohiro, and Alexis Akira Toda. 2024a. Bubble Economics. *Journal of Mathematical Economics* 111: 102944.
Hirano, Tomohiro, and Alexis Akira Toda. 2024b. Note on Bubbles Attached to Real Assets. Working paper, October 22. **Link**
Hirano, Tomohiro, and Alexis Akira Toda. 2024c. Rational Bubbles: A Clarification. Working paper, July 19. **Link**
Hirano, Tomohiro, and Alexis Akira Toda. 2025. Bubble Necessity Theorem. *Journal of Political Economy* 133(1): 111–145.
Hori, Takeo, Ryonghun Im, and Hiroshi Nakaota. 2024. Bubbly Fundamentals. Discussion Paper No. 278. School of Economics, Kwansei Gakuin University (Nishinomiya, Japan). **Link**
Ikeda, Daisuke. 2022. Monetary Policy, Inflation, and Rational Asset Price Bubbles. *Journal of Money, Credit and Banking* 54(6): 1569–1603.
Jordà, Òscar, Moritz Schularick, and Alan M. Taylor. 2015. Leveraged Bubbles. *Journal of Monetary Economics* 76: S1–S20.
Kocherlakota, Narayana R. 1992. Bubbles and Constraints on Debt Accumulation. *Journal of Economic Theory* 57(1): 245–256.
Kocherlakota, Narayana R. 2008. Injecting Rational Bubbles. *Journal of Economic Theory* 142(1): 218–232.
Lagos, Ricardo, Guillaume Rocheteau, and Randall Wright. 2017. Liquidity: A New Monetarist Perspective. *Journal of Economic Literature* 55(2): 371–440.
Lewis, C. S. 1960. *The Four Loves*. Harcourt Brace.
Magill, Michael, and Martine Quinzii. 1994. Infinite Horizon Incomplete Markets. *Econometrica* 62(4): 853–880.
Magill, Michael, and Martine Quinzii. 1996. Incomplete Markets Over an Infinite Horizon: Long-Lived Securities and Speculative Bubbles. *Journal of Mathematical Economics* 26(1): 133–170.
Martin, Alberto, and Jaume Ventura. 2018. The Macroeconomics of Rational Bubbles: A

User's Guide. *Annual Review of Economics* 10: 505–539.

Miao, Jianjun. 2014. *Economic Dynamics in Discrete Time*. MIT Press.

Miao, Jianjun, and Pengfei Wang. 2012. Bubbles and Total Factor Productivity. *American Economic Review: Papers and Proceedings* 102(3): 82–87.

Miao, Jianjun, and Pengfei Wang. 2014. Sectoral Bubbles, Misallocation, and Endogenous Growth. *Journal of Mathematical Economics* 53(3): 153–163.

Miao, Jianjun, and Pengfei Wang. 2015. Banking Bubbles and Financial Crises. *Journal of Economic Theory* 157(3): 763–792.

Miao, Jianjun, and Pengfei Wang. 2018 (MW). Asset Bubbles and Credit Constraints. *American Economic Review* 108(9): 2590–2628.

Miao, Jianjun, Pengfei Wang, and Lifang Xu. 2016. Stock Market Bubbles and Unemployment. *Economic Theory* 61: 273–307.

Miao, Jianjun, Pengfei Wang, and Zhiwei Xu. 2015. A Bayesian Dynamic Stochastic General Equilibrium Model of Stock Market Bubbles and Business Cycles. *Quantitative Economics* 6(3): 599–635.

Montrucchio, Luigi. 2004. Cass Transversality Condition and Sequential Asset Bubbles. *Economic Theory* 24(3): 645–663.

Olivier, Jacques. 2000. Growth-Enhancing Bubbles. *International Economic Review* 41(1): 133–152.

Phillips, Peter C. B., and Shuping Shi. 2020. Real Time Monitoring of Asset Markets: Bubbles and Crises. In *Handbook of Statistics*, vol. 42, ed. Hrishikesh D. Vinod and C. R. Rao, 61–80. Elsevier.

Phillips, Peter C. B., Shuping Shi, and Jun Yu. 2015. Testing for Multiple Bubbles: Historical Episodes of Exuberance and Collapse in the S&P 500. *International Economic Review* 56(4): 1043–1078.

Samuelson, Paul A. 1958. An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. *Journal of Political Economy* 66(6): 467–482.

Santos, Manuel S., and Michael Woodford. 1997. Rational Asset Pricing Bubbles. *Econometrica* 65(1): 19–57.

Scheinkman, José A., and Lawrence Weiss. 1986. Borrowing Constraints and Aggregate Economic Activity. *Econometrica* 54(1): 23–45.

Scheinkman, José A., and Wei Xiong. 2003. Overconfidence and Speculative Bubbles. *Journal of Political Economy* 111(6): 1183–1220.

Sorger, Gerhard. 2020. On the Dynamics of Stock Price Bubbles: Comments on a Model by Miao and Wang. *Central European Journal of Operations Research* 28(2): 521–537.

Stiglitz, Joseph E. 1990. Symposium on Bubbles. *Journal of Economic Perspectives* 4(2): 13–18.

Tirole, Jean. 1985. Asset Bubbles and Overlapping Generations. *Econometrica* 53(6): 1499–1528.

Toda, Alexis Akira. 2025. *Essential Mathematics for Economics*. CRC Press.

Wilson, Charles A. 1981. Equilibrium in Dynamic Models with an Infinity of Agents. *Journal of Economic Theory* 24(1): 95–111.

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Go to archive of Comments section
Go to March 2025 issue