Discuss this article at Journaltalk:
http://journaltalk.net/articles/5939

EJW
ECON JOURNAL WATCH
Scholarly Comments on Academic Economics

Econ Journal Watch 14(2)
May 2017: 133–137

The Chang-Kim-Park Model of Cointegrated Density-Valued Time Series Cannot Accommodate a Stochastic Trend

Brendan K. Beare[1]

Link to Abstract

In their article "Nonstationarity in Time Series of State Densities," published in the *Journal of Econometrics* in 2016, Yoosoon Chang, Chang Sik Kim, and Joon Y. Park—referred to hereafter as CKP—propose a model of nonstationary cointegrated dynamics for a time series of probability density functions (hereafter "pdfs"). The main novelty of their approach is that at each point in time, the value taken by the time series in question is not a point in finite dimensional Euclidean space but rather a function, in fact a pdf, which we can view as an element of an infinite-dimensional vector space. While this necessarily complicates some of the mathematics, at an intuitive level their model can be understood to be an infinite-dimensional version of the usual model of cointegration. CKP fit their model to a time series of cross-sectional distributions of individual earnings, and a time series of intra-month distributions of stock returns.

Unfortunately, the dynamic model proposed by CKP has a rather serious flaw: Any time series of pdfs that satisfies their assumptions must necessarily be stationary, aside from a deterministic component. In particular, a stochastic trend driving long-run variation in their time series of pdfs—the defining feature of cointegration—cannot be present within their framework. The problem arises

1. University of California–San Diego, La Jolla, CA 92093.

from the nonnegativity property of pdfs, a property that turns out to be incompatible with the postulated form of nonstationarity.

To build some intuition for what is going on, suppose we have two independent and identically distributed ("iid") sequences of random variables, $(x_t, t \in \mathbb{N})$ and $(y_t, t \in \mathbb{N})$, which are independent of one another, with each x_t and y_t equal to 1 with probability ½ and equal to −1 with probability ½. Construct a third sequence of random variables $(z_t, t \in \mathbb{N})$ by setting

$$z_t = a + b\sum_{s=1}^{t} x_s + y_t \tag{1}$$

for each $t \in \mathbb{N}$, for some constants $a, b \in \mathbb{R}$. It should be clear that the time series (z_t) is stationary (in fact iid) if $b = 0$, and nonstationary if $b \neq 0$. But what if I told you that each random variable z_t is guaranteed to be nonnegative? For z_t to be a nonnegative random variable we require that $a \geq |b|t + 1$. This can only be true for all $t \in \mathbb{N}$ if in fact $b = 0$, and so we see that the requirement of nonnegativity forces (z_t) to be stationary.

The situation in CKP's paper is more complicated than in the simple example just provided, but it is the same tension between nonnegativity and nonstationarity that leads to the problem with their model. The problem is obscured by the fact that CKP work with demeaned pdfs. Though a demeaned pdf need not be nonnegative, it must nevertheless be bounded from below by a function that integrates to minus one. We will see that this leads to the same problem.

What follows is necessarily somewhat technical. I encourage readers who wish to learn more about time series analysis in infinite dimensional spaces to consult the monograph of Denis Bosq (2000).

Let K denote a compact interval of real numbers, and let B denote the Banach space of Lebesgue equivalence classes of integrable real valued functions on K equipped with the norm $\|f\|_1 = \int_K |f|$. CKP's time series is a sequence of pdf-valued random elements $(f_t, t \in \mathbb{N}_0)$ of B, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $t \in \mathbb{N}_0$ we let $w_t = f_t - \mathrm{E}f_t$, with the expectation operator E defined as a Bochner integral in the usual way. We may informally view $\mathrm{E}f_t$ as a pointwise Lebesgue integral $\mathrm{E}f_t(x)$ taken for each $x \in K$, though strictly speaking $f_t(x)$ is not well-defined for fixed $x \in K$ as f_t is a Lebesgue equivalence class of functions.

Let H denote the Hilbert space of Lebesgue equivalence classes of square integrable real-valued functions on K that integrate to zero, with inner product $\langle f, g\rangle = \int_K fg$ and corresponding norm $\|\cdot\|_2$. CKP assume that $(w_t, t \in \mathbb{N}_0)$ is a sequence of random elements of H. For $t \in \mathbb{N}$ we let $u_t = w_t - w_{t-1}$. CKP assume that there is an iid sequence $(\varepsilon_t, t \in \mathbb{Z})$ in H with $\mathrm{E}\|\varepsilon_0\|_2^2 < \infty$ and positive definite covariance operator Σ such that, for each $t \in \mathbb{N}$, we have

$$u_t = \sum_{k=0}^{\infty} \Phi_k(\varepsilon_{t-k}).$$

Here $(\Phi_k, k \in \mathbb{N}_0)$ is a sequence in $\mathscr{L}(H)$, the space of bounded linear operators from H to H. The sequence of operators is assumed to satisfy the summability condition $\sum_{k=0}^{\infty} k\|\Phi_k\|_{\mathscr{L}(H)} < \infty$, where $\|\cdot\|_{\mathscr{L}(H)}$ is the operator norm. Under this summability condition, CKP observe that a version of the Beveridge-Nelson decomposition applies: We have

$$w_t = w_0 - \tilde{u}_0 + \sum_{s=1}^{t} \Phi(\varepsilon_s) + \tilde{u}_t \tag{2}$$

for each $t \in \mathbb{N}$, where $\Phi = \sum_{k=0}^{\infty} \Phi_k$, and $(\tilde{u}_t, t \in \mathbb{Z})$ is the stationary sequence of random elements of H defined by $\tilde{u}_t = \sum_{k=0}^{\infty} \widetilde{\Phi}_k(\varepsilon_{t-k})$, with $\widetilde{\Phi}_k = -\sum_{j=k+1}^{\infty} \Phi_j$.

Proposition: Under the conditions just stated, $\Phi = 0$.

Before diving into the proof of this Proposition, let's recall the very simple example used earlier to illustrate the tension between nonnegativity and nonstationarity. Compare equation (1) to the Beveridge-Nelson decomposition (2). We can think of z_t as playing the role of the demeaned pdf w_t, of a as playing the role of the initial condition $w_0 - \tilde{u}_0$, of b as playing the role of the operator Φ, of x_s as playing the role of the innovation ε_s, and of y_t as playing the role of the stationary component $\tilde{u}_t$. Just as a nonnegativity condition on z_t forced b to be equal to zero in our simple example, the proof of the above Proposition will demonstrate that the requirement that w_t be bounded from below by a function that integrates to minus one—specifically, by the negative of $\mathrm{E}f_t$—forces Φ to be equal to zero.

Proof of Proposition: Each f_t is pdf-valued and hence nonnegative, so in view of the Beveridge-Nelson decomposition (2) we must have

$$w_0 - \tilde{u}_0 + \sum_{s=1}^{t} \Phi(\varepsilon_s) + \tilde{u}_t \geq -\mathrm{E}f_t \tag{3}$$

for all $t \in \mathbb{N}$. Fix an arbitrary Borel set $J \subseteq K$. A consequence of (3) is that we must have

$$\tfrac{1}{\sqrt{t}}\int_J \mathrm{E}f_t + \tfrac{1}{\sqrt{t}}\int_J (w_0 - \tilde{u}_0) + \int_J \tfrac{1}{\sqrt{t}}\sum_{s=1}^{t} \Phi(\varepsilon_s) + \tfrac{1}{\sqrt{t}}\int_J \tilde{u}_t \geq 0 \tag{4}$$

for all $t \in \mathbb{N}$. Consider the behavior of each of the four terms on the left-hand side of the inequality (4) as $t \to \infty$. The Bochner integral $\mathrm{E}f_t$ is nonnegative and satisfies $\int_K \mathrm{E}f_t = \mathrm{E}\int_K f_t = 1$, so we have $0 \leq \int_J \mathrm{E}f_t \leq 1$ and hence $t^{-1/2}\int_J \mathrm{E}f_t \to 0$. Further, since $\int_J (w_0$

$- \tilde{u}_0$) and $\int_J \tilde{u}_t$ are random variables whose law does not depend on t—the latter by stationarity—we have $t^{-1/2}\int_J(w_0 - \tilde{u}_0) \to 0$ and $t^{-1/2}\int_J \tilde{u}_t \to 0$ in probability. The central limit theorem for Hilbertian random elements (Bosq 2000, Theorem 2.7) ensures that $t^{-1/2}\sum_{s=1}^{t} \Phi(\varepsilon_s)$ converges in law to Z, a Gaussian random element of H with covariance operator $\Phi\Sigma\Phi^*$, where Φ^* is the adjoint to Φ. It then follows from the continuous mapping theorem that $\int_J t^{-1/2}\sum_{s=1}^{t} \Phi(\varepsilon_s) \to \int_J Z$ in law. Putting these results together with Slutsky's theorem, we find that

$$\frac{1}{\sqrt{t}}\int_J \mathrm{E}f_t + \frac{1}{\sqrt{t}}\int_J\left(w_0 - \tilde{u}_0\right) + \int_J \frac{1}{\sqrt{t}}\sum_{s=1}^{t} \Phi\left(\varepsilon_s\right) + \frac{1}{\sqrt{t}}\int_J \tilde{u}_t \ \to \int_J Z \tag{5}$$

in law as $t \to \infty$. Now, $\int_J Z$ is a centered Gaussian random variable, so the only way that (4) and (5) can simultaneously be true is if $\int_J Z$ has variance zero, so that $\int_J Z = 0$ almost surely. That is, $\int_K Z 1_J = 0$ almost surely, where 1_J denotes the indicator function for J. This is true for every Borel set $J \subseteq K$ so, appealing to the fact that every element of H is the limit of a sequence of simple functions on K, we have $\langle Z, h\rangle = 0$ almost surely for every $h \in H$. Recall that the covariance operator of Z is $\Phi\Sigma\Phi^*$; applying the definition of a covariance operator, we obtain $\Phi\Sigma\Phi^*(h) = \mathrm{E}(\langle Z, h\rangle Z) = 0$ for all $h \in H$, so that $\Phi\Sigma\Phi^* = 0$. Since Σ is positive definite, this proves that $\Phi = 0$. ■

With $\Phi = 0$, the random-walk component of the Beveridge-Nelson decomposition vanishes, and for each $t \in \mathbb{N}$ we have $f_t = w_0 - \tilde{u}_0 + \mathrm{E}f_t + \tilde{u}_t$. CKP's time series of pdfs is thus the sum of a deterministic component $w_0 - \tilde{u}_0 + \mathrm{E}f_t$ (deterministic in the sense of being determined at time zero) and a stationary component $\tilde{u}_t$.

The flaw elaborated above with the CKP model derives from the fact that the usual L^2 Hilbert space is poorly suited to handling pdf-valued random variables, an insight that has been noted in prior mathematical literature. The literature has developed suitable Hilbert spaces in which addition and scalar multiplication are defined such that they preserve the essential properties of pdfs (see, e.g., Egozcue, Díaz-Barrero, and Pawlowsky-Glahn 2006; van den Boogaart, Egozcue, and Pawlowsky-Glahn 2014; Petersen and Müller 2016). I thank Peter Phillips and Won-Ki Seo for pointing me to such references, and for very helpful discussions of how the structures developed therein might form the basis for more useful models of cointegrated pdf-valued time series. Further research in this direction is underway.

References

Bosq, Denis. 2000. *Linear Processes in Function Spaces*. New York: Springer.

Chang, Yoosoon, Chang Sik Kim, and Joon Y. Park. 2016. Nonstationarity in Time Series of State Densities. *Journal of Econometrics* 192(1): 152–167.

Egozcue, Juan José, José Luis Díaz-Barrero, and Vera Pawlowsky-Glahn. 2006. Hilbert Space of Probability Density Functions Based on Aitchison Geometry. *Acta Mathematica Sinica* 22(4): 1175–1182.

Petersen, Alexander, and Hans-Georg Müller. 2016. Functional Data Analysis for Density Functions by Transformation to a Hilbert Space. *Annals of Statistics* 44(1): 183–218.

van den Boogaart, Karl Gerald, Juan José Egozcue, and Vera Pawlowsky-Glahn. 2014. Bayes Hilbert Spaces. *Australian & New Zealand Journal of Statistics* 56(2): 171–194.

About the Author

Brendan K. Beare is Associate Professor of Economics at the University of California–San Diego. He has published a range of articles on theoretical and applied econometrics in outlets including *Econ Journal Watch*, *Econometrica*, *Econometric Theory*, the *Journal of Applied Econometrics*, and the *Journal of Time Series Analysis*. His email address is bbeare@ucsd.edu.

Go to archive of Comments section
Go to May 2017 issue

Discuss this article at Journaltalk:
http://journaltalk.net/articles/5939